

# Analytical Models of the Proton Structure Function and Gluon Distributions at Small $x$ Beyond Leading Order

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**Abstract.** We incorporate the next-to-leading order (NLO) and the next-to-next-to-leading order (NNLO) effects in the models of the Singlet Structure function  $F_2^S(x, t)$  and the gluon distribution  $G(x, t)$  using DGLAP equations approximated at small  $x$ . Analytical solutions of Singlet Structure function both at the next-to leading order (NLO) as well as the next-to-next-to leading order (NNLO) are obtained. We also make phenomenological study of our result comparing with the exact results.

KEY WORDS: DGLAP equations, Gluon distributions, Singlet structure function.

## 1 Introduction

Study of Proton structure function at small  $x$  is an important area of research in recent years. Recently we have reported an analysis of the proton structure function as well as the gluon distribution at small  $x$  using the Taylor approximated [1,2] DGLAP equations. The precision of the recent experimental data [7] demands the correction terms of the splitting function atleast upto NLO [8,9] and preferably NNLO [10–12] in DGLAP evolution equations. In the present paper we obtain the corresponding  $t$  evolution of the structure function both at NLO and NNLO. In order to obtain their analytical forms we use plausible relationship between the singlet and gluon distributions [13, 20, 21] and use Lagrange's Auxiliary method [14] to solve the corresponding first order partial differential equation in  $x$  and  $t$ . In Section 2 we discuss the formalism, Section 3 is devoted to numerical analysis and lastly in Section 4 we give our conclusions.

## 2 Formalism

### 2.1 Taylor approximated coupled DGLAP equations at small $x$ in NLO and NNLO

The standard forms of coupled DGLAP evolution equations for the singlet and the gluon distributions are given as [3, 5, 6]

$$\frac{\partial F_2^S(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 dz \left[ P_{qq}(z, \alpha_s(Q^2)) F_2^S\left(\frac{x}{z}, Q^2\right) + 2N_f P_{qg}(z, \alpha_s(Q^2)) G\left(\frac{x}{z}, Q^2\right) \right] \quad (1)$$

$$\frac{\partial G(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 dz \left[ P_{gg}(z, \alpha_s(Q^2)) G\left(\frac{x}{z}, Q^2\right) + P_{gq}(z, \alpha_s(Q^2)) F_2^S\left(\frac{x}{z}, Q^2\right) \right], \quad (2)$$

where  $N_f$  is the flavor number.  $F_2^S(x, Q^2) = x\Sigma(x, Q^2)$  is the singlet structure function,  $\Sigma(x, Q^2) = \sum_{i=1}^{N_f} (q_i(x, Q^2) + \bar{q}_i(x, Q^2))$  is the singlet quark distribution and  $q_i$  and  $\bar{q}_i$  are the quarks and antiquarks of flavour  $i$ .  $G(x, Q^2) = xg(x, Q^2)$  is the gluon distribution function. The splitting functions  $P'_{ij}$ s are Altarelli-Parisi splitting kernels and are calculable in the perturbative approach to QCD. They have been known, for a long time, at three-loop accuracy Ref. [10, 11], which is the next-to-next-to-leading order (NNLO or N2LO) in the expansion in powers of the strong coupling  $\alpha_s$

$$P_{ij}(x, \alpha_s(Q^2)) = P_{ij}^{LO}(x) + \frac{\alpha_s(Q^2)}{2\pi} P_{ij}^{NLO}(x) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2 P_{ij}^{NNLO}(x). \quad (3)$$

The explicit expressions for  $P_{ij}^{LO}$  are as in Ref. [6, 15, 16],  $P_{ij}^{NLO}$  as in Ref. [8, 9] and  $P_{ij}^{NNLO}$  as in Ref. [10–12, 17].

The scale ( $Q^2$ ) dependence of the strong coupling is controlled by the  $\beta$ -function which can be expressed in perturbative series. The one loop (LO), two loop (NLO) and three loop (NNLO) solutions of the running coupling constant  $\alpha_s/2\pi$  are respectively [18]

$$\frac{\alpha_s^{LO}(t)}{2\pi} = \frac{2}{\beta_0 t} \quad (4)$$

$$\frac{\alpha_s^{NLO}(t)}{2\pi} = \frac{2}{\beta_0 t} \left[ 1 - \frac{\beta_1 \ln t}{\beta_0^2 t} \right] \quad (5)$$

$$\frac{\alpha_s^{NNLO}(t)}{2\pi} = \frac{2}{\beta_0 t} \left[ 1 - \frac{\beta_1 \ln t}{\beta_0^2 t} + \frac{1}{(\beta_0 t)^2} \left( \frac{\beta_1^2}{\beta_0^2} (\ln^2 t - \ln t + 1) + \frac{\beta_2}{\beta_0} \right) \right], \quad (6)$$

where

$$\beta_0 = \frac{1}{3}(33 - 2N_f), \quad \beta_1 = 102 - \frac{38}{3}N_f, \quad \beta_2 = \frac{2857}{6} - \frac{6673}{18}N_f + \frac{325}{54}N_f^2$$

are the one-loop, two-loop and three-loop corrections respectively to the QCD  $\beta$ -function and  $t$  is defined as  $t = \ln(Q^2/\Lambda^2)$  and  $\Lambda$  is the QCD cut-off scale parameter.

The Taylor approximation of  $F_2^S(\frac{x}{z}, Q^2)$  and gluon  $G(\frac{x}{z}, Q^2)$  upto  $O(x)$  [1, 2]

$$F_2^S\left(\frac{x}{z}, Q^2\right) = F_2^S(x, Q^2) + x \frac{(1-z)}{z} F_2^S(x, Q^2) \quad (7)$$

$$G\left(\frac{x}{z}, Q^2\right) = G(x, Q^2) + x \frac{(1-z)}{z} G(x, Q^2) \quad (8)$$

Substituting the splitting functions upto NLO and up to NNLO in the DGLAP equations, we obtain the Taylor approximated DGLAP equations [1, 2] for the singlet and gluon distribution as the following:

For NLO:

$$\begin{aligned} \frac{\partial F_2^S(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \int_x^1 dz \left[ P_{qq}^{LO} F_2^S(x, t) + P_{qq}^{LO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right. \\ &\quad \left. + 2N_f \left( P_{qg}^{LO} G(x, t) + P_{qg}^{LO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x} \right) \right] \\ &\quad + \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \int_x^1 dz \left[ P_{qq}^{NLO} F_2^S(x, t) + P_{qq}^{NLO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right. \\ &\quad \left. + 2N_f \left( P_{qg}^{NLO} G(x, t) + P_{qg}^{NLO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x} \right) \right], \quad (9) \end{aligned}$$

$$\begin{aligned} \frac{\partial G(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \int_x^1 dz \left[ P_{gg}^{LO} G(x, t) + P_{gg}^{LO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x} \right. \\ &\quad \left. + \left( P_{gq}^{LO} F_2^S(x, t) + P_{gq}^{LO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right) \right] \\ &\quad + \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \int_x^1 dz \left[ P_{gg}^{NLO} G(x, t) + P_{gg}^{NLO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x} \right. \\ &\quad \left. + P_{gq}^{NLO} F_2^S(x, t) + P_{gq}^{NLO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right]. \quad (10) \end{aligned}$$

For NNLO:

$$\begin{aligned}
\frac{\partial F_2^S(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \int_x^1 dz \left[ P_{qq}^{LO} F_2^S(x, t) + P_{qq}^{LO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right. \\
&\quad \left. + 2N_f (P_{qg}^{LO} G(x, t) + P_{qg}^{LO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x}) \right] \\
&\quad + \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \int_x^1 dz \left[ P_{qq}^{NLO} F_2^S(x, t) + P_{qq}^{NLO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right. \\
&\quad \left. + 2N_f (P_{qg}^{NLO} G(x, t) + P_{qg}^{NLO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x}) \right] \\
&\quad + \left( \frac{\alpha_s(t)}{2\pi} \right)^3 \int_x^1 dz \left[ P_{qq}^{NNLO} F_2^S(x, t) + P_{qq}^{NNLO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right. \\
&\quad \left. + 2N_f (P_{qg}^{NNLO} G(x, t) + P_{qg}^{NNLO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x}) \right], \quad (11)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial G(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \int_x^1 dz \left[ P_{gg}^{LO} G(x, t) + P_{gg}^{LO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x} \right. \\
&\quad \left. + (P_{gq}^{LO} F_2^S(x, t) + P_{gq}^{LO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x}) \right] \\
&\quad + \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \int_x^1 dz \left[ P_{gg}^{NLO} G(x, t) + P_{gg}^{NLO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x} \right. \\
&\quad \left. + P_{gq}^{NLO} F_2^S(x, t) + P_{gq}^{NLO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right] \\
&\quad + \left( \frac{\alpha_s(t)}{2\pi} \right)^3 \int_x^1 dz \left[ P_{gg}^{NNLO} G(x, t) + P_{gg}^{NNLO} x \frac{(1-z)}{z} \frac{\partial G(x, t)}{\partial x} \right. \\
&\quad \left. + P_{gq}^{NNLO} F_2^S(x, t) + P_{gq}^{NNLO} x \frac{(1-z)}{z} \frac{\partial F_2^S(x, t)}{\partial x} \right]. \quad (12)
\end{aligned}$$

Taylor approximated DGLAP eqns. (9), (10), (11) and (12) can be solved only if one assumes plausible analytical relationship between the singlet and gluon distribution. Generally, exact analytical solution of coupled DGLAP equations or an explicit relation between quark and gluon distributions are not possible. Numerical methods are necessary. However, if one assumes that at small  $x$  in a certain  $Q^2$  range, such analytical form of gluon and quark distribution is possible, the most general form can be written as

$$G(x, t) = xg(x, t) = x \sum_i^{2N_f} K_i C_i(x, t) \{q_i(x, t) + \bar{q}_i(x, t)\}, \quad (13)$$

which takes into account the flavor independency of gluon. If the coefficients  $K_i$  and  $C_i(x, t)$  are flavor independent then the expression can be written as

$$G(x, t) = KC(x, t)F_2^S(x, t). \quad (14)$$

Here,  $KC(x, t)$  represents the ratio of the quark and gluon distribution, i.e  $\frac{G(x, t)}{F_2^S(x, t)}$  and is in general not factorizable in  $x$  and  $t$ . A purely  $t$ -independence of the ratio was found to be not true in general [19]. In conformity with the QCD analysis of Lopez and Yndurain [13] and pursued by us later in [20, 21] we use the plausible  $t$  dependent relationship between the singlet and the gluon distribution

$$G(x, t) = Kt^\sigma F_2^S(x, t), \quad (15)$$

where  $K$  and  $\sigma$  are fitted from experiments [2].

Next-to leading order (NLO):

The above eqns. (9) and (10) becomes respectively as

$$\begin{aligned} \frac{\partial F_2^S(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \left[ A_1^S(x, t) F_2^S(x, t) + B_1^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right] \\ &+ \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \left[ C_1^S(x, t) F_2^S(x, t) + D_1^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right], \quad (16) \end{aligned}$$

$$\begin{aligned} \frac{\partial F_2^S(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \left[ A_2^S(x, t) F_2^S(x, t) + B_2^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right] \\ &+ \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \left[ C_2^S(x, t) F_2^S(x, t) + D_2^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right], \quad (17) \end{aligned}$$

where  $A_{1,2}^S(x, t)$ ,  $B_{1,2}^S(x, t)$ ,  $C_{1,2}^S(x, t)$ ,  $D_{1,2}^S(x, t)$  are the integrals over splitting functions as given in the eqns. (A1)-(A8) of Appendix A. The Lagrange's auxiliary method [14] can be applied to solve eqns.(16) and (17) analytically only if the  $t$  evolution of the strong coupling constant at NLO  $(\alpha_s^{NLO}(t))^2$  can be linearised. Defining  $T(t) = \alpha_s(t)/2\pi$ , we linearise it to be  $\left( \frac{\alpha_s(t)}{2\pi} \right)^2 = T_1 T(t) \implies T^2(t) = T_1 T(t)$  [22–24], where  $T_1$  is a parameter to be determined from the particular range of  $Q^2$  range under study. Following Lagrange's method [14] to solve eqns. (16) and (17), we put them in the form as,

$$Q_1^{NLO}(x, t) \frac{\partial F_2^S(x, t)}{\partial t} + P_1^{NLO}(x, t) \frac{\partial F_2^S(x, t)}{\partial x} = R_1^{NLO}(x, t) F_2^S(x, t), \quad (18)$$

$$Q_2^{NLO}(x, t) \frac{\partial F_2^S(x, t)}{\partial t} + P_2^{NLO}(x, t) \frac{\partial F_2^S(x, t)}{\partial x} = R_2^{NLO}(x, t) F_2^S(x, t), \quad (19)$$

where

$$Q_1^{NLO}(x, t) = 1, \quad (20)$$

$$P_1^{NLO}(x, t) = -T(t)[B_1^S(x, t) + T_1 D_1^S(x, t)], \quad (21)$$

$$R_1^{NLO}(x, t) = T(t)[A_1^S(x, t) + T_1 C_1^S(x, t)] \quad (22)$$

and

$$Q_2^{NLO}(x, t) = 1, \quad (23)$$

$$P_2^{NLO}(x, t) = -T(t)[B_2^S(x, t) + T_1 D_2^S(x, t)], \quad (24)$$

$$R_2^{NLO}(x, t) = T(t)[A_2^S(x, t) + T_1 C_2^S(x, t)]. \quad (25)$$

The Lagrange's equations (18) and (19) are obtained from the solutions of the auxiliary equation

$$\frac{dx}{P_{1,2}^{NLO}(x, t)} = \frac{dt}{1} = \frac{dF_2^S(x, t)}{R_{1,2}^{NLO}(x, t)F_2^S(x, t)}. \quad (26)$$

The general solution of equations (18) and (19) is given by

$$f(u_{1,2}^{NLO}, v_{1,2}^{NLO}) = 0, \quad (27)$$

where  $f(u_{1,2}^{NLO}, v_{1,2}^{NLO})$  is arbitrary function of  $u_{1,2}^{NLO}, v_{1,2}^{NLO}$  are defined below in eqns. (28) and (30). Let  $u_{1,2}^{NLO}(x, t) = C_1'$  and  $v_{1,2}^{NLO}(x, t, F_2^S(x, t)) = D_1'$  be two independent solutions of eqn. (26). Solving eqn. (27), we obtain

$$u_{1,2}^{NLO}(x, t) = tX_{1,2}^{NLO}(x, t) \quad (28)$$

$$v_{1,2}^{NLO}(x, t, F_2^S(x, t)) = F_2^S(x, t)Y_{1,2}^{NLO}(x, t), \quad (29)$$

where

$$X_{1,2}^{NLO}(x, t) = t^{b/t} e^{b/t} \exp \left[ \frac{1}{a} \int \frac{dx}{B_{1,2}^S(x, t) + T_1 D_{1,2}^S(x, t)} \right], \quad (30)$$

$$Y_{1,2}^{NLO}(x, t) = \exp \left[ \int \frac{A_{1,2}^S(x, t) + T_1 C_{1,2}^S(x, t)}{B_{1,2}^S(x, t) + T_1 D_{1,2}^S(x, t)} dx \right] \quad (31)$$

and  $a = 2/\beta_0$ ;  $b = \beta_1/\beta_0^2$ .

The linear combination of  $u_{1,2}^{NLO}(x, t)$  and  $v_{1,2}^{NLO}(x, t, F_2^S(x, t))$  in  $F_2^S(x, t)$  as in [1]

$$u_{1,2}^{NLO} + \alpha v_{1,2}^{NLO} = \beta^{NLO} \quad (32)$$

gives

$$F_2^{S(I,II),NLO}(x, t) = \frac{1}{\alpha Y_{1,2}^{NLO}(x, t)} [\beta - tX_{1,2}^{NLO}(x, t)] \quad (33)$$

$F_2^{S(I),NLO}(x, t)$  and  $F_2^{S(II),NLO}(x, t)$  are the solutions of eqns. (18) and (19) respectively. Using the boundary condition at certain  $t = t_0$ ,

$$F_2^{S(I),NLO}(x, t_0) = F_2^{S(II),NLO}(x, t_0) = F_2^S(x, t_0) \quad (34)$$

We obtain two alternative  $t$ - evolution equations for Singlet distribution analytically at NLO as

$$F_2^{S(I),NLO}(x,t) = F_2^S(x,t_0) \frac{t}{t_0} \left( \frac{Y_1^{NLO}(x,t_0)}{Y_1^{NLO}(x,t)} \right) \left[ \frac{X_1^{NLO}(x,t) - \frac{\beta}{t}}{X_1^{NLO}(x,t_0) - \frac{\beta}{t_0}} \right] \quad (35)$$

$$F_2^{S(II),NLO}(x,t) = F_2^S(x,t_0) \frac{t}{t_0} \left( \frac{Y_2^{NLO}(x,t_0)}{Y_2^{NLO}(x,t)} \right) \left[ \frac{X_2^{NLO}(x,t) - \frac{\beta}{t}}{X_2^{NLO}(x,t_0) - \frac{\beta}{t_0}} \right] \quad (36)$$

with ratio

$$\begin{aligned} R^{NLO}(x,t) &= \frac{F_2^{S(I),NLO}(x,t)}{F_2^{S(II),NLO}(x,t)} \\ &= \frac{\left( \frac{Y_1^{NLO}(x,t_0)}{Y_1^{NLO}(x,t)} \right) \left[ \frac{X_1^{NLO}(x,t) - \beta/t}{X_1^{NLO}(x,t_0) - \beta/t_0} \right]}{\left( \frac{Y_2^{NLO}(x,t_0)}{Y_2^{NLO}(x,t)} \right) \left[ \frac{X_2^{NLO}(x,t) - \beta/t}{X_2^{NLO}(x,t_0) - \beta/t_0} \right]}, \end{aligned} \quad (37)$$

which is not equal to unity in general as in LO [1, 2]. Even if the factor  $\beta$  vanishes, because of the  $t$  dependence of the functions  $X_1^{NLO}(x,t)$ ,  $Y_1^{NLO}(x,t)$ ,  $X_2^{NLO}(x,t)$ ,  $Y_2^{NLO}(x,t)$ , the ratio will not be identity [1, 2].

The integral functions of  $X_1^{NLO}(x,t)$ ,  $Y_1^{NLO}(x,t)$ ,  $X_2^{NLO}(x,t)$ ,  $Y_2^{NLO}(x,t)$  occurring in eqns. (35) and (36) are as follows:

$$\begin{aligned} X_1^{NLO}(x,t) &= t^{bt} e^{bt} \exp \left[ \frac{1}{a} \int dx \left( a_1(K,t,\sigma) + b_1(K,t,\sigma)x \right. \right. \\ &\quad \left. \left. + c_1(K,t,\sigma)x \ln(1/x) \right)^{-1} \right], \end{aligned} \quad (38)$$

$$\begin{aligned} Y_1^{NLO}(x,t) &= \exp \left[ \int dx \left( a_2(K,t,\sigma) + b_2(K,t,\sigma)x + d_2(K,t,\sigma) \ln(1/x) \right) \right. \\ &\quad \left. \times \left( a_1(K,t,\sigma) + b_1(K,t,\sigma)x + c_1(K,t,\sigma)x \ln(1/x) \right)^{-1} \right], \end{aligned} \quad (39)$$

$$\begin{aligned} X_2^{NLO}(x,t) &= t^{b/t} e^{b/t} \exp \left[ \frac{1}{a} \int dx \left( a_3(K,t,\sigma) + b_3(K,t,\sigma)x \right. \right. \\ &\quad \left. \left. + c_3(K,t,\sigma)x \ln(1/x) + d_3(K,t,\sigma) \ln(1/x) \right)^{-1} \right], \end{aligned} \quad (40)$$

$$\begin{aligned} Y_2^{NLO}(x,t) &= \exp \left[ \int dx \left( (a_4(K,t,\sigma) + b_4(K,t,\sigma)x + d_4(K,t,\sigma) \ln(1/x)) \right. \right. \\ &\quad \left. \left. \times \left( a_3(K,t,\sigma) + b_3(K,t,\sigma)x + c_3(K,t,\sigma)x \ln(1/x) \right. \right. \right. \\ &\quad \left. \left. \left. + d_3(K,t,\sigma) \ln(1/x) \right)^{-1} \right]. \end{aligned} \quad (41)$$

The coefficients  $a_1(K,t,\sigma)$ ,  $b_1(K,t,\sigma)$ ,  $c_1(K,t,\sigma)$ ,  $a_2(K,t,\sigma)$ ,  $b_2(K,t,\sigma)$ ,

$d_2(K, t, \sigma)$ ,  $a_3(K, t, \sigma)$ ,  $b_3(K, t, \sigma)$ ,  $c_3(K, t, \sigma)$ ,  $d_3(K, t, \sigma)$ ,  $a_4(K, t, \sigma)$ ,  $b_4(K, t, \sigma)$ ,  $d_4(K, t, \sigma)$  are as given in [Appendix B](#).

Next-to-next-to leading order (NNLO):

For next-to-next-to leading order, substituting eqn. (15) in the the evolution equations for the singlet and gluon distributions eqns. (11) and (12) respectively becomes

$$\begin{aligned} \frac{\partial F_2^S(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \left[ A_1^S(x, t) F_2^S(x, t) + B_1^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right] \\ &+ \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \left[ C_1^S(x, t) F_2^S(x, t) + D_1^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right] \\ &+ \left( \frac{\alpha_s(t)}{2\pi} \right)^3 \left[ L_1^S(x, t) F_2^S(x, t) + M_1^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right], \quad (42) \end{aligned}$$

$$\begin{aligned} \frac{\partial F_2^S(x, t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi} \left[ A_2^S(x, t) F_2^S(x, t) + B_2^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right] \\ &+ \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \left[ C_2^S(x, t) F_2^S(x, t) + D_2^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right] \\ &+ \left( \frac{\alpha_s(t)}{2\pi} \right)^3 \left[ L_2^S(x, t) F_2^S(x, t) + M_2^S(x, t) \frac{\partial F_2^S(x, t)}{\partial x} \right], \quad (43) \end{aligned}$$

where  $L_{1,2}^S(x, t)$ ,  $M_{1,2}^S(x, t)$  are as given in the eqns. (A9)-(A12) of [Appendix A](#).

To proceed further and solve the above two eqns. (42) and (43) by Lagrange's method, we need to linearize the cubic term of the strong coupling constant at NNLO. We linearise through the ansatz  $T^3(t) = T_2^2 T(t)$ , where  $T(t) = \alpha_s(t)/2\pi$  [22–24] and  $T_2$  is a suitable parameter to be determined from the particular range of  $Q^2$  range under study. Solving the following Lagrange's equations:

$$Q_1^{NNLO} \frac{\partial F_2^S(x, t)}{\partial t} + P_1^{NNLO}(x, t) \frac{\partial F_2^S(x, t)}{\partial x} = R_1^{NNLO}(x, t) F_2^S(x, t), \quad (44)$$

$$Q_2^{NNLO} \frac{\partial F_2^S(x, t)}{\partial t} + P_2^{NNLO}(x, t) \frac{\partial F_2^S(x, t)}{\partial x} = R_2^{NNLO}(x, t) F_2^S(x, t), \quad (45)$$

where

$$Q_{1,2}^{NNLO}(x, t) = 1, \quad (46)$$

$$P_{1,2}^{NNLO}(x, t) = -T(t) [B_{1,2}^S(x, t) + T_1 D_{1,2}^S(x, t) + T_2^2 M_{1,2}^S(x, t)], \quad (47)$$

$$R_{1,2}^{NNLO}(x, t) = T(t) [A_{1,2}^S(x, t) + T_1 C_{1,2}^S(x, t) + T_2^2 L_{1,2}^S(x, t)], \quad (48)$$

we obtain

$$u_{1,2}^{NNLO}(x, t) = t X_{1,2}^{NNLO}(x, t), \quad (49)$$

$$v_{1,2}^{NNLO}(x, t, F_2^S(x, t)) = F_2^S(x, t) Y_{1,2}^{NNLO}(x, t), \quad (50)$$

where

$$X_{1,2}^{NNLO}(x, t) = t^{b/t} \exp\left(\frac{b}{t} - \frac{b^2 \ln^2 t + b^2 + c}{2t^2}\right) \times \exp\left[\frac{1}{a} \int \frac{dx}{B_{1,2}^S(x, t) + T_1 D_{1,2}^S(x, t) + T_2^2 M_{1,2}^S(x, t)}\right] \quad (51)$$

$$Y_{1,2}^{NNLO}(x, t) = \exp\left[\int \frac{A_{1,2}^S(x, t) + T_1 C_{1,2}^S(x, t) + T_2^2 L_{1,2}^S(x, t)}{B_{1,2}^S(x, t) + T_1 D_{1,2}^S(x, t) + T_2^2 M_{1,2}^S(x, t)} dx\right]. \quad (52)$$

Here  $a = 2\beta_0$ ;  $b = \beta_1/\beta_0^2$  as introduced after eqn. (31) and  $c = \beta_2/\beta_0^3$ .

The linear combination of  $u_{1,2}^{NNLO}(x, t)$  and  $v_{1,2}^{NNLO}(x, t, F_2^S(x, t))$  in  $F_2^S(x, t)$  as in [1]

$$u_{1,2}^{NNLO} + \alpha v_{1,2}^{NNLO} = \beta^{NNLO} \quad (53)$$

gives two alternative  $t$  evolution equation for Singlet distribution in NNLO as

$$F_2^{S(I),NNLO}(x, t) = F_2^S(x, t_0) \frac{t}{t_0} \left(\frac{Y_1^{NNLO}(x, t_0)}{Y_1^{NNLO}(x, t)}\right) \times \left[\frac{X_1^{NNLO}(x, t) - \beta/t}{X_1^{NNLO}(x, t_0) - \beta/t_0}\right] \quad (54)$$

$$F_2^{S(II),NNLO}(x, t) = F_2^S(x, t_0) \frac{t}{t_0} \left(\frac{Y_2^{NNLO}(x, t_0)}{Y_2^{NNLO}(x, t)}\right) \times \left[\frac{X_2^{NNLO}(x, t) - \beta/t}{X_2^{NNLO}(x, t_0) - \beta/t_0}\right] \quad (55)$$

with ratio

$$R^{NNLO}(x, t) = \frac{F_2^{S(I),NNLO}(x, t)}{F_2^{S(II),NNLO}(x, t)} = \frac{\left(\frac{Y_1^{NNLO}(x, t_0)}{Y_1^{NNLO}(x, t)}\right) \left[\frac{X_1^{NNLO}(x, t) - \beta/t}{X_1^{NNLO}(x, t_0) - \beta/t_0}\right]}{\left(\frac{Y_2^{NNLO}(x, t_0)}{Y_2^{NNLO}(x, t)}\right) \left[\frac{X_2^{NNLO}(x, t) - \beta/t}{X_2^{NNLO}(x, t_0) - \beta/t_0}\right]}, \quad (56)$$

which is not equal to unity in general as discussed in LO [1,2] and NLO because of the  $t$  dependence of  $X_1^{NNLO}(x, t)$  and  $X_2^{NNLO}(x, t)$  even if  $\beta$  is set to zero. It is to be noted that the term  $\beta$  occurring in eqns. (35), (36) and (54), (55) at LO, NLO and NNLO may not be in general identical i.e  $\beta^{LO} \neq \beta^{NLO} \neq \beta^{NNLO}$ .

The equations (35), (36) and (54), (55) are our main analytical expressions for singlet distributions in NLO and NNLO respectively. The explicit expressions for  $X_{1,2}^{NNLO}(x, t)$ ,  $Y_{1,2}^{NNLO}(x, t)$  occurring eqns. (54) and (55) are as follows:

$$X_1^{NNLO}(x, t) = t^{b/t} \exp\left(\frac{b}{t} - \frac{b^2(\ln t)^2 + b^2 + c}{2t^2}\right) \times \exp\left[\frac{1}{a} \int \frac{dx}{a_5(K, t, \sigma) + b_5(K, t, \sigma)x + c_5(K, t, \sigma)x \ln(1/x)}\right], \quad (57)$$

$$Y_1^{NNLO}(x, t) = \exp \left[ \int dx \left( (a_6(K, t, \sigma) + b_6(K, t, \sigma)x + d_6(K, t, \sigma) \ln(1/x)) \right) \left( a_5(K, t, \sigma) + b_5(K, t, \sigma)x + c_5(K, t, \sigma)x \ln(1/x) \right)^{-1} \right], \quad (58)$$

$$X_2^{NNLO}(x, t) = t^{b/t} \exp \left( \frac{b}{t} - \frac{b^2(\ln t)^2 + b^2 + c}{2t^2} \right) \exp \left[ \frac{1}{a} \int dx \left( a_7(K, t, \sigma) + b_7(K, t, \sigma)x + c_7(K, t, \sigma)x \ln(1/x) + d_7(K, t, \sigma) \ln(1/x) \right)^{-1} \right], \quad (59)$$

$$Y_2^{NNLO}(x, t) = \exp \left[ \int dx \left( a_8(K, t, \sigma) + b_8(K, t, \sigma)x + d_8(K, t, \sigma) \ln(1/x) \right) \left( a_7(K, t, \sigma) + b_7(K, t, \sigma)x + c_7(K, t, \sigma)x \ln(1/x) + d_7(K, t, \sigma) \ln(1/x) \right)^{-1} \right]. \quad (60)$$

The coefficients  $a_5(K, t, \sigma)$ ,  $b_5(K, t, \sigma)$ ,  $c_5(K, t, \sigma)$ ,  $a_6(K, t, \sigma)$ ,  $b_6(K, t, \sigma)$ ,  $d_6(K, t, \sigma)$ ,  $a_7(K, t, \sigma)$ ,  $b_7(K, t, \sigma)$ ,  $c_7(K, t, \sigma)$ ,  $d_7(K, t, \sigma)$ ,  $a_8(K, t, \sigma)$ ,  $b_8(K, t, \sigma)$ ,  $d_8(K, t, \sigma)$  are as given in [Appendix B](#).

A structure of these functions indicate that they are not analytically solvable.

## 2.2 Approximate analytical expressions for structure functions

### 2.2.1 At Leading order (LO)

For numerical analysis of the leading order we use our previous result as given in the following equations of Ref [2]

$$F_2^{S(I),LO}(x, t) = F_2^S(x, t_0) \left( \frac{t}{t_0} \right) \frac{Y_1^{LO}(x, t_0)}{Y_1^{LO}(x, t)} \left[ \frac{X_1^{LO}(x, t) - \frac{\beta}{t}}{X_1^{LO}(x, t_0) - \frac{\beta}{t_0}} \right], \quad (61)$$

$$F_2^{S(II),LO}(x, t) = F_2^S(x, t_0) \left( \frac{t}{t_0} \right) \frac{Y_2^{LO}(x, t_0)}{Y_2^{LO}(x, t)} \left[ \frac{X_2^{LO}(x, t) - \frac{\beta}{t}}{X_2^{LO}(x, t_0) - \frac{\beta}{t_0}} \right], \quad (62)$$

where the analytical forms of  $X_1^{LO}(x, t)$ ,  $Y_1^{LO}(x, t)$ ,  $X_2^{LO}(x, t)$  and  $Y_2^{LO}(x, t)$  are as the following:

$$X_1^{LO}(x, K, t, \sigma) = \exp \left[ - \frac{\ln(\ln \frac{1}{x})}{A_f(2 + \frac{3N_f}{2} K t^\sigma)} \right] \quad (63)$$

$$Y_1^{LO}(x, K, t, \sigma) = \exp \left[ -\frac{1}{(2 + \frac{3}{2}N_f K t^\sigma)} \left( \int \frac{(2 - \frac{3}{2}N_f K t^\sigma)}{\ln \frac{1}{x}} dx + \int \frac{N_f K t^\sigma}{x \ln \frac{1}{x}} dx \right) \right], \quad (64)$$

$$X_2^{LO}(x, K, t, \sigma) = \exp \left[ \frac{K t^\sigma}{9A_f} \frac{x}{(K t^\sigma + \frac{4}{9})} \right], \quad (65)$$

$$Y_2^{LO}(x, K, t, \sigma) = \exp \left[ \frac{-K t^\sigma (9 + \frac{4}{K t^\sigma})}{9(K t^\sigma + \frac{4}{9})} x \ln \frac{1}{x} \right] \quad (66)$$

and  $A_f = 4/(3\beta_0)$

From the phenomenological observation of both  $F_2^{S(I),LO}(x, t)$  and  $F_2^{S(II),LO}(x, t)$  as in Ref [2], we obtain  $F_2^{S(I),LO}(x, t)$  to be more preferred than the other because of the range of validity in  $x$  and  $Q^2$  [1] and hence we make the further analysis of NLO and NNLO only with the first evolutions  $F_2^{S(I),NLO}(x, t)$  and  $F_2^{S(I),NNLO}(x, t)$ .

### 2.2.2 At next-to-leading order (NLO) and at next-to-next-to-leading order (NNLO)

The analytical forms of the structure functions as in eqns. (35) at NLO and (54) at NNLO are possible only if we make additional assumptions below whose validity will be tested subsequently.

We observe that at NLO, among the three functions  $a_1(K, t, \sigma)$ ,  $b_1(K, t, \sigma)$  and  $c_1(K, t, \sigma)$  arising in eqn.(35) as given in Appendix B, for a given  $K, t, \sigma$ , the function  $a_1(K, t, \sigma)$  is smaller compared to  $b_1(K, t, \sigma)$  and  $c_1(K, t, \sigma)$ , in the denominators of  $X_1^{NLO}(x, t)$  and  $Y_1^{NLO}(x, t)$  of eqns.(38) and (39) respectively. Under this assumption  $X_1^{NLO}(x, t)$  and  $Y_1^{NLO}(x, t)$  are obtained as the following:

$$X_1^{NLO}(x, t) = t^{b/t} e^{b/t} \exp \left[ \frac{1}{a} \left( \frac{-1}{b_1} \log \frac{1}{x} + \frac{c_1}{b_1^2} \frac{(\log \frac{1}{x})^2}{2} \right) \right] \quad (67)$$

$$Y_1^{NLO}(x, t) = \exp \left[ \left( \frac{b_2}{b_1} - \frac{b_2 c_1}{b_1^2} \right) x - \frac{a_2}{b_1} \log \frac{1}{x} - \left( \frac{d_2}{b_1} - \frac{a_2 c_1}{b_1^2} \right) \frac{(\log \frac{1}{x})^2}{2} + \frac{d_2 c_1}{b_1^2} \frac{(\log \frac{1}{x})^3}{3} - \frac{b_2 c_1}{b_1^2} x \log \frac{1}{x} \right], \quad (68)$$

where  $a = 2/\beta_0$ ;  $b = \beta_1/\beta_0^2$  as introduced after eqn. (31) and  $c = \beta_2/\beta_0^3$  and the coefficients  $b_1, c_1, a_2, b_2, d_2$  are as given in Appendix B.

Hence

$$F_2^{S(I),NLO}(x, t) = F_2^S(x, t_0) \frac{t}{t_0} \left( \frac{Y_1^{NLO}(x, t_0)}{Y_1^{NLO}(x, t)} \right) \left[ \frac{X_1^{NLO}(x, t) - \frac{\beta}{t}}{X_1^{NLO}(x, t_0) - \frac{\beta}{t_0}} \right]. \quad (69)$$

Similarly at NNLO, among the three functions  $b_5(K, t, \sigma)$ ,  $a_5(K, t, \sigma)$  and  $c_5(K, t, \sigma)$  arising in eqn. (54) as given in Appendix B in the denominators of  $X_1^{NNLO}$  and  $Y_1^{NNLO}$  of eqns. (57) and (58) respectively for a given  $K, t, \sigma$  the function  $b_5(K, t, \sigma)$  is smaller compared to  $a_5(K, t, \sigma)$  and  $c_5(K, t, \sigma)$ . Under this assumption  $X_1^{NNLO}(x, t)$  and  $Y_1^{NNLO}(x, t)$  are obtained as the following:

$$X_1^{NNLO}(x, t) = t^{b/t} \exp\left(\frac{b}{t} - \frac{b^2(\ln t)^2 + b^2 + c}{2t^2}\right) \times \exp\left[\frac{1}{a}\left(\frac{x}{a_5} - \frac{c_5 x^2}{a_5^2 4} - \frac{c_5 x^2}{a_5^2 2} \log \frac{1}{x}\right)\right], \quad (70)$$

$$Y_1^{NNLO}(x, t) = \exp\left[x(a_6 + d_6) - x^2\left(\frac{1}{4} \frac{a_6 c_5}{a_5} - \frac{b_6}{2} + \frac{1}{4} \frac{c_5 d_6}{a_5}\right) + \frac{x^3}{9} \frac{b_6 c_5}{a_5} + x \log \frac{1}{x} d_6 + x^2 \log \frac{1}{x} \left(-\frac{a_6 c_5}{2a_5} - \frac{c_5 d_6}{2a_5}\right) - x^3 \log\left(\frac{1}{x}\right) \frac{b_6 c_5}{3a_5} - x^2 \left(\log \frac{1}{x}\right)^2 \frac{c_5 d_6}{3a_5}\right], \quad (71)$$

where the coefficients  $a_5, c_5, a_6, b_6, d_6$  are as given in Appendix B.

Hence

$$F_2^{S(I), NNLO}(x, t) = F_2^S(x, t_0) \frac{t}{t_0} \frac{Y_1^{NNLO}(x, t_0)}{Y_1^{NNLO}(x, t)} \frac{X_1^{NNLO}(x, t) - \frac{\beta}{t}}{X_1^{NNLO}(x, t_0) - \frac{\beta}{t_0}}. \quad (72)$$

Equations (35) and (54) with the definitions of  $X_1^{NLO}(x, t)$ ,  $Y_1^{NLO}(x, t)$  and  $X_1^{NNLO}(x, t)$ ,  $Y_1^{NNLO}(x, t)$  as given in eqns. (69) and (72) are the equations to be tested in the following numerical section.

### 3 Numerical Analysis

Before testing the models with data and exact result let us first make some observation on the parameters  $T_1$  and  $T_2$  occurred in the present approach at NLO eqns.(35) and NNLO eqns.(54)

NLO:

From the above equations (30) and (31), it is noticed that the evolution  $F_2^{S(I)NLO}$  contains a parameter  $T_1$  ( $< 1$ ) which does not occur at LO. It is, therefore, reasonable to relate  $T_1$  as the average value of the QCD running coupling constant for the  $Q^2$  range under study.

$$T_1 \propto \frac{\langle \alpha_s(t)^{NLO} \rangle}{2\pi} = c^{NLO} \frac{1}{2\pi(t_2 - t_1)} \int_{t_1}^{t_2} \alpha_s(t)^{NLO} dt. \quad (73)$$

Here  $c^{NLO}$  is the proportionality constant and is first set as unity. Choosing  $t_1$  and  $t_2$  in the  $Q$  range  $2 \leq Q \leq 100$ , we obtain  $T_1 = 0.02704$  from eqn. (73). To

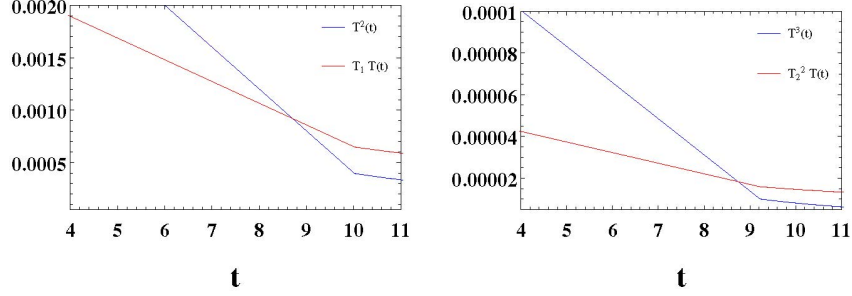


Figure 1. Plot of  $T^2(t)^{NLO}$  and  $T_1 T(t)^{LO}$  (left) and plot of  $T^3(t)^{NNLO}$  and  $T_2^2 T(t)^{LO}$  (right) versus  $t$  for their corresponding  $c^{NLO}$  and  $c^{NNLO}$  taken to be unity.

study the qualitative feature of the linearised relation, we plot  $T^2(t)$  and  $T_1 T(t)$  as a function of  $t$  as in Figure 1 (left). The figure indicates that the linearised approximation  $T_1 T(t)$  appears to be valid at around the intersection point of  $t = 8.8$ , corresponding to  $Q^2 = 590 \text{ GeV}^2$ .

NNLO:

Similarly for NNLO, the linearized parameter  $T_2 (< 1)$  occurs in the formalism in eqns. (51) and (52). For a range between  $t_1 = \ln(Q_1^2/\Lambda^2)$  to  $t_2 = \ln(Q_2^2/\Lambda^2)$ ,

$$T_2 \propto \frac{\langle \alpha_s(t)^{NNLO} \rangle}{2\pi} = c^{NNLO} \frac{1}{2\pi(t_2 - t_1)} \int_{t_1}^{t_2} \alpha_s(t)^{NNLO} dt. \quad (74)$$

Again the proportionality constant  $c^{NNLO}$  is first set as unity. Choosing  $t_1$  and  $t_2$  in the  $Q$  range  $2 \leq Q \leq 100$ , we obtain  $T_2 = 0.0247$  from eqn. (74). To study the qualitative feature of the linearised relation, we plot Figure 1 (right), and it shows that the linearisation approximation is valid at around small range of  $t = 8.8$ , i.e  $Q^2 = 590 \text{ GeV}^2$ .

We, therefore, first check the validity of our model around  $Q^2 = 590 \text{ GeV}^2$  where the NLO and NNLO conditions are valid.

In the previous work [2], the range of validity of the model was  $2 \times 10^{-6} \leq x \leq 1 \times 10^{-3}$  and  $2 \leq Q \leq 100 \text{ GeV}$  for  $\beta = -1.016$ ,  $K = 1.856$ ,  $\sigma = 0.227$ . Using the same set of values for those parameters we make our analysis at NLO and NNLO effects first with the obtained  $T_1 = 0.02704$  and  $T_2 = 0.0247$  respectively in the same  $x$  and  $Q^2$  range.

In Figure 2, we plot the singlet structure Functions for LO  $F_2^{S(I),LO}$  (61); for NLO  $F_2^{S(I),NLO}$  (69) and for NNLO  $F_2^{S(I),NNLO}$  (72), for a range of small  $x$ :  $2 \times 10^{-6} \leq x \leq 1 \times 10^{-3}$  and at fixed  $Q^2 = 590 \text{ GeV}^2$  and then compare with the NNPDF3.0 data [28]. The orange, black and red solid lines in the

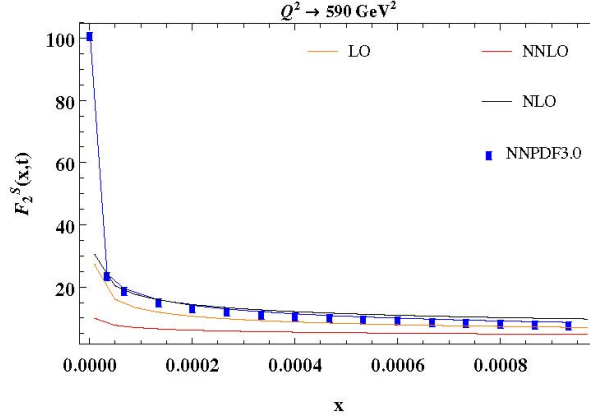


Figure 2. Plots of  $F_2^{S(I),LO}$  and  $F_2^{S(I),NLO}$ ,  $F_2^{S(I),NNLO}$  as a function of  $x$  for fixed  $Q^2 = 590 \text{ GeV}^2$  and comparison with the NNPDF3.0 data.

figure represents the LO, NLO and NNLO evolutions respectively. We observe that the LO as well as the NLO effects are much closer to data. But when NNLO effects are added the prediction comes below the data. For instance it is observed that varying the values of  $T_1$  and  $T_2$  bring the prediction closer to data. As an illustration we show for  $T_1 = 0.5$  and  $T_2 = 10^{-4}$  with the corresponding values of  $C^{NLO} = 18.491$  and  $C^{NNLO} = 0.004$  as in Figure 3. At those values of  $T_1$  and  $T_2$  the NLO prediction shows a high consistency with the data but we do not observe any significant rise in the evolution for NNLO correction, which

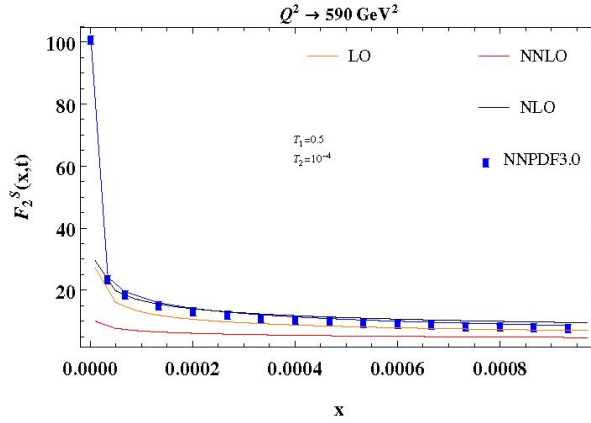


Figure 3. Plots of  $F_2^{S(I),LO}$  and  $F_2^{S(I),NLO}$ ,  $F_2^{S(I),NNLO}$  as a function of  $x$  for fixed  $Q^2 = 590 \text{ GeV}^2$  using  $T_1 = 0.5$  and  $T_2 = 10^{-4}$  and comparison with the NNPDF3.0 data.

interprets that the parameter  $T_2$  is having a very minimal contribution in the evolution for NNLO correction.

Figure 4 represents the results of  $Q$  evolution of the singlet structure Functions  $F_2^{S(I),LO}$  and  $F_2^{S(I),NLO}$ ,  $F_2^{S(I),NNLO}$  for a few values of small  $x$  bins. Using our initial set of parameters  $T_1 = 0.02704$  and  $T_2 = 0.0247$  in our models  $F_2^{S(I),NLO}$  (35) and  $F_2^{S(I),NNLO}$  (54), we compare with the NNPDF3.0 data [28]. In this figure, the orange, black and red solid lines represents LO, NLO and NNLO evolutions respectively. We observe that our model upto NLO is valid but when NNLO corrections are taken then the structure function deviates much below the data as well as with the LO and NLO result. This shows that our small  $x$  approximations in NLO model is validated than those in our NNLO model. As observed from this work it is clear that the region of validity of our

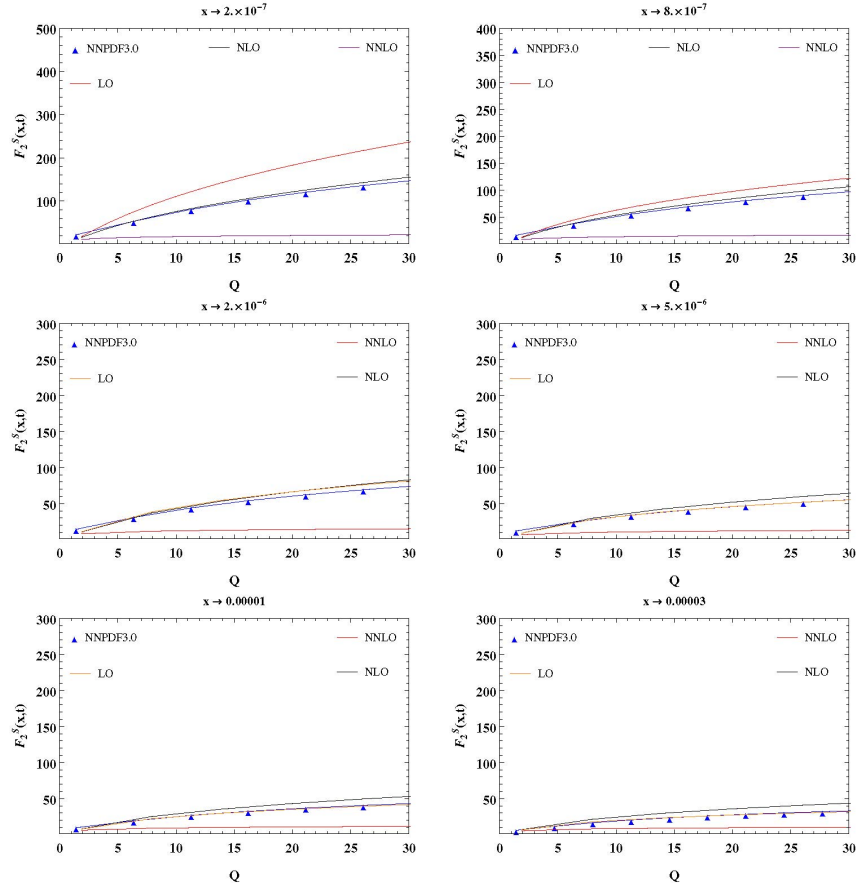


Figure 4. Plots of  $F_2^{S(I),LO}$  and  $F_2^{S(I),NLO}$ ,  $F_2^{S(I),NNLO}$  as a function of  $Q$  for fixed  $x$  and comparison with the NNPDF3.0 [28].

method in NLO model is approximately in the region towards smaller  $x$  range:  $10^{-7} \leq x \leq 10^{-4}$  for fixed  $Q^2 \approx 590 \text{ GeV}^2$ .

#### 4 Conclusions

In this paper, we obtained the approximate analytical expressions for the singlet structure functions both at NLO and NNLO from the Taylor approximated coupled DGLAP equations assuming the adhoc  $t$  dependent relation between the singlet and the gluon distribution consistent with the analysis of Lopez and Yndurain [13, 21]. At LO the phenomenological analysis is done with the three sets of parameters  $\beta = -1.016$ ,  $K = 1.856$ ,  $\sigma = 0.227$  while for NLO and NNLO additional parameters  $T_1 = 0.02704$  and  $T_2 = 0.0247$  respectively are needed. Our analysis are done around  $Q^2 = 590 \text{ GeV}^2$  which corresponds to our approximation where our higher order approximation is valid. At the extreme limit of  $T_2 = 0$  theoretically, we recover the NLO result from NNLO (as can be observed from eqn. (51) and (52)) but we do not obtain it from the analytical expression ((70) and (71)), which may predict that the various approximations assumed in solving the DGLAP equations upto NNLO may not be all justified.

However we observe that our analytical solution for evolutions of singlet structure function are in good consistency with the experimental data upto NLO. With our present model upto NLO, the phenomenological range of validity for a fixed  $Q^2 \approx 590 \text{ GeV}^2$ , is much towards the region of smaller  $x$  range:  $10^{-7} \leq x \leq 10^{-4}$ , unlike the previous LO model [1, 2]. Though various methods like Laguerre polynomials, Mellin transformation etc. are available in literature in order to obtain a numerical solution of DGLAP evolution equations, our method to solve these equations analytically is also a workable alternative upto NLO.

Finally we conclude this work with a few comments regarding its limitation. There are several approximations which are used to analytically solve the DGLAP equations at small  $x$  upto NNLO with valid degree of justification:

(i) The PDFs are expanded at small  $x$  (7) which has inherent theoretical limitations. In DGLAP evolutions, the integrals extends from 0 to 1. Thus at small  $x$  one cannot neglect in principle the medium/large  $x$  behaviour of the Parton Distribution functions.

(ii) Equation (15) relating the singlet and the Gluon distribution assumes the  $x$  dependence of both the distribution to be the same upto a proportionality factor, which may not be true in general. At small scale ( $Q^2$ ) the standard PDF available in current literature Ref. [34] suggest such difference. Even in ultra small  $x$  and ultra high  $Q^2$  double asymptotic scaling limit suggest different  $x$  dependence for the singlet and gluon distribution. Specifically at double asymptotic limit (ultra small  $x$  and ultra large  $Q^2$ ) the corresponding gluon and the singlet distribution have different  $x$  dependence in general [2, 35, 36]. Thus the present assumption appears to be approximately true at limited  $x$  and  $Q^2$  as has been observed in

the present work.

(iii) In analytical solutions beyond LO, we need to linearise the  $t$ -dependence of the coupling constant  $\alpha^{NLO}$  (eqn.(5)) and  $\alpha^{NNLO}$  (eqn.(6)). It introduces the parameter  $T_1$  and  $T_2$  at NLO and NNLO respectively, which is the inherent limitation of the present analytical approach as had been noted in earlier communication [24]. Such approximations are however found to be true in the limited  $Q^2$  around 590 GeV<sup>2</sup>.

### Appendix A

The expressions for the functions  $A_i^S(x, t)$ ,  $B_i^S(x, t)$ ,  $C_i^S(x, t)$ ,  $D_i^S(x, t)$ ,  $L_i^S(x, t)$ ,  $M_i^S(x, t)$ , where  $i = 1, 2$  involving the integral representation containing the splitting functions as occurred in eqns. (16), (17) and (42), (43) are as follows.  $A_i^S(x, t)$ ,  $B_i^S(x, t)$ ,  $C_i^S(x, t)$ ,  $D_i^S(x, t)$  are same in both the cases of NLO and NNLO.

$$A_1^S(x, t) = \int_x^1 dz \left[ P_{qq}^{LO}(x) + 2N_f P_{qg}^{LO}(x) Kt^\sigma \right], \quad (A1)$$

$$B_1^S(x, t) = \int_x^1 dz \left[ P_{qq}^{LO}(x) \frac{x(1-z)}{z} + 2N_f P_{qg}^{LO}(x) \frac{x(1-z)}{z} Kt^\sigma \right], \quad (A2)$$

$$C_1^S(x, t) = \int_x^1 dz \left[ P_{qq}^{NLO}(x) + 2N_f P_{qg}^{NLO}(x) Kt^\sigma \right], \quad (A3)$$

$$D_1^S(x, t) = \int_x^1 dz \left[ P_{qq}^{NLO}(x) \frac{x(1-z)}{z} + 2N_f P_{qg}^{NLO}(x) \frac{x(1-z)}{z} Kt^\sigma \right], \quad (A4)$$

$$A_2^S(x, t) = \frac{1}{Kt^\sigma} \int_x^1 dz \left[ P_{gg}^{LO}(x) Kt^\sigma + P_{gq}^{LO}(x) \right], \quad (A5)$$

$$B_2^S(x, t) = \frac{1}{Kt^\sigma} \int_x^1 dz \left[ P_{gg}^{LO}(x) \frac{x(1-z)}{z} Kt^\sigma + P_{gq}^{LO}(x) \frac{x(1-z)}{z} \right], \quad (A6)$$

$$C_2^S(x, t) = \frac{1}{Kt^\sigma} \int_x^1 dz \left[ P_{gg}^{NLO}(x) Kt^\sigma + P_{gq}^{NLO}(x) \right], \quad (A7)$$

$$D_2^S(x, t) = \frac{1}{Kt^\sigma} \int_x^1 dz \left[ P_{gg}^{NLO}(x) \frac{x(1-z)}{z} Kt^\sigma + P_{gq}^{NLO}(x) \frac{x(1-z)}{z} \right] \quad (A8)$$

and

$$L_1^S(x, t) = \int_x^1 dz \left[ P_{qq}^{NNLO}(x) + 2N_f P_{qg}^{NNLO}(x) Kt^\sigma \right], \quad (A9)$$

$$M_1^S(x, t) = \int_x^1 dz \left[ P_{qq}^{NNLO}(x) \frac{x(1-z)}{z} + 2N_f P_{qg}^{NNLO}(x) \frac{x(1-z)}{z} Kt^\sigma \right], \quad (A10)$$

$$L_2^S(x, t) = \frac{1}{Kt^\sigma} \int_x^1 dz \left[ P_{gg}^{NNLO}(x) Kt^\sigma + P_{gq}^{NNLO}(x) \right], \quad (A11)$$

$$M_2^S(x, t) = \frac{1}{Kt^\sigma} \int_x^1 dz \left[ P_{gg}^{NNLO}(x) \frac{x(1-z)}{z} Kt^\sigma + P_{gq}^{NNLO}(x) \frac{x(1-z)}{z} \right]. \quad (A12)$$

**Appendix B**

The coefficients  $a_1(K, t, \sigma)$ ,  $b_1(K, t, \sigma)$ ,  $c_1(K, t, \sigma)$ ,  $a_2(K, t, \sigma)$ ,  $b_2(K, t, \sigma)$ ,  $d_2(K, t, \sigma)$ ,  $a_3(K, t, \sigma)$ ,  $b_3(K, t, \sigma)$ ,  $c_3(K, t, \sigma)$ ,  $d_3(K, t, \sigma)$ ,  $a_4(K, t, \sigma)$ ,  $b_4(K, t, \sigma)$ ,  $d_4(K, t, \sigma)$  as given in eqns. (38)-(41) and the coefficients  $a_5(K, t, \sigma)$ ,  $b_5(K, t, \sigma)$ ,  $c_5(K, t, \sigma)$ ,  $a_6(K, t, \sigma)$ ,  $b_6(K, t, \sigma)$ ,  $d_6(K, t, \sigma)$ ,  $a_7(K, t, \sigma)$ ,  $b_7(K, t, \sigma)$ ,  $c_7(K, t, \sigma)$ ,  $d_7(K, t, \sigma)$ ,  $a_8(K, t, \sigma)$ ,  $b_8(K, t, \sigma)$ ,  $d_8(K, t, \sigma)$  as given in eqns. (57)-(60) are listed below:

$$a_1(K, t, \sigma, T_1) = T_1 \left( \frac{80}{27} N_f + \frac{20}{3} K t^\sigma \right) \quad (\text{B1})$$

$$a_2(K, t, \sigma, T_1) = \frac{8}{3} + \frac{2N_f K t^\sigma}{3} + T_1 \left[ -435.136 - \frac{384217N_f}{810} + K t^\sigma \left( \frac{16\pi^2 + 649}{216} N_f + \frac{1219597}{1080} \right) \right] + T_1 \left( -3414 + \frac{23}{6} K t^\sigma \right) \ln(2) \quad (\text{B2})$$

$$a_3(K, t, \sigma, T_1) = 6 + \frac{8}{3Kt^\sigma} + T_1 \left[ \frac{3843}{48} - \frac{61}{9} N_f - 3\pi^2 + \frac{1}{Kt^\sigma} \left( \frac{517}{18} - \frac{4}{3} \pi^2 - \frac{80}{27} N_f \right) \right] \quad (\text{B3})$$

$$a_4(K, t, \sigma, T_1) = \frac{11}{12} - \frac{N_f}{3} - 11 - \frac{12}{Kt^\sigma} - T_1 \left( \frac{20035}{3601} + \frac{7\pi^2}{2} - \frac{679705.51}{1320} \right) + \frac{1}{Kt^\sigma} \left( \frac{80}{27} N_f + \frac{11\pi^2}{3} + \frac{2317}{864} + \frac{103837.66}{81} \right) + T_1 \left( \frac{45}{2} \right) \quad (\text{B4})$$

$$a_5(K, t, \sigma, T_2) = T_2 \left( \frac{80N_f}{27} + \frac{20Kt^\sigma}{3} \right) + T_2^2 \left( -19666.936 + 38600.48Kt^\sigma \right) \quad (\text{B5})$$

$$a_6(K, t, \sigma, T_2) = \frac{8}{3} + \frac{2N_f K t^\sigma}{3} + T_2 \left( -435.136 - \frac{384217N_f}{810} \right) + T_2 \left( -3414 + \frac{23}{6} K t^\sigma \right) \ln(2) + T_2^2 \left( 4061.071 - 3921.587 \times 8Kt^\sigma \right) \quad (\text{B6})$$

$$a_7(K, t, \sigma, T_2) = 6 + \frac{8}{3Kt^\sigma} + T_2 \left[ \frac{3843}{48} - \frac{61N_f}{9} - 3\pi^2 + \frac{1}{Kt^\sigma} \left( \frac{517}{18} - \frac{4\pi^2}{3} - \frac{80N_f}{27} \right) \right] + T_2^2 \left( 135983.174 + \frac{3240.749}{Kt^\sigma} \right) \quad (\text{B7})$$

$$\begin{aligned}
a_8(K, t, \sigma, T_1) &= \frac{11}{12} - \frac{N_f}{3} - 11 - \frac{12}{Kt^\sigma} \\
&\quad - T_1 \left[ \frac{20035}{3601} + \frac{7\pi^2}{2} - \frac{679705.51}{1320} \right. \\
&\quad \left. + \frac{1}{Kt^\sigma} \left( \frac{80}{27} N_f + \frac{11\pi^2}{3} + \frac{2317}{864} + \frac{103837.66}{81} \right) \right. \\
&\quad \left. + T_1 \left( \frac{45}{2} \right) \right] + T_2^2 \left( -87979.883 - \frac{28540.98}{Kt^\sigma} \right) \quad (B8)
\end{aligned}$$

$$\begin{aligned}
b_1(K, t, \sigma, T_1) &= -\frac{5}{3} N_f Kt^\sigma + T_1 \left[ \frac{416151}{324} N_f - \frac{627740.73}{5832} \right. \\
&\quad \left. + Kt^\sigma \left( \frac{1659953}{2160} + \frac{\pi^2}{27} \right) + \left( -\frac{204}{81} - \frac{883}{4} Kt^\sigma \right) \ln(2) \right] \quad (B9)
\end{aligned}$$

$$b_2(K, t, \sigma, T_1) = \frac{4}{3} - N_f Kt^\sigma + T_1 \left( \frac{-16791.29}{12} + \frac{3593 N_f}{108} + \frac{125 Kt^\sigma}{16} \right) \quad (B10)$$

$$\begin{aligned}
b_4(K, t, \sigma, T_1) &= \frac{1}{Kt^\sigma} (12Kt^\sigma + 16) + T_1 \left[ \frac{679}{108} N_f - \frac{2305}{32} - 4\pi^2 \right. \\
&\quad \left. - \frac{1}{Kt^\sigma} \left( \frac{79649}{81} - \frac{4\pi^2}{3} - \frac{128}{27} N_f \right) \right] \quad (B11)
\end{aligned}$$

$$\begin{aligned}
b_5(K, t, \sigma, T_2) &= -\frac{5}{3} N_f Kt^\sigma + T_2 \left[ \frac{416151 N_f}{324} - \frac{627740.73}{5832} \right. \\
&\quad \left. + Kt^\sigma \left( \frac{1659953}{2160} + \frac{\pi^2}{27} \right) \right] + T_2 \left( -\frac{204}{81} \ln(2) - \frac{883}{4} Kt^\sigma \ln(2) \right) \\
&\quad + T_2^2 (-12075.005 - 329142.4 Kt^\sigma) \quad (B12)
\end{aligned}$$

$$\begin{aligned}
b_6(K, t, \sigma, T_2) &= \frac{4}{3} - N_f Kt^\sigma + T_2 \left( \frac{-16791.29}{12} + \frac{3593 N_f}{108} + \frac{125 Kt^\sigma}{16} \right) \\
&\quad + T_2^2 (-2143.669 - 22467.4 \times 8 Kt^\sigma) \quad (B13)
\end{aligned}$$

$$\begin{aligned}
b_7(K, t, \sigma, T_2) &= \left( 8 + \frac{2}{3 Kt^\sigma} \right) + T_2 \left( \frac{989}{108} N_f + 4\pi^2 + \frac{2557}{32} - \frac{20}{Kt^\sigma} \right) \\
&\quad + T_2 \left[ \frac{1}{Kt^\sigma} \left( \frac{5104}{81} + 4\pi^2 + \frac{208}{27} N_f \right) \right] \\
&\quad + T_2^2 \left( -139453.698 - \frac{22436.898}{Kt^\sigma} \right) \quad (B14)
\end{aligned}$$

$$\begin{aligned}
b_8(K, t, \sigma, T_2) &= \frac{1}{Kt^\sigma} (12Kt^\sigma + 16) + T_2 \left[ \frac{679}{108} N_f - \frac{2305}{32} - 4\pi^2 \right. \\
&\quad \left. - \frac{1}{Kt^\sigma} \left( \frac{79649}{81} - \frac{4\pi^2}{3} - \frac{128}{27} N_f \right) \right] \\
&\quad + T_2^2 \left( 49919.486 + \frac{19159.5}{Kt^\sigma} \right) \quad (B15)
\end{aligned}$$

(B16)

$$c_1(K, t, \sigma, T_1) = T_1 \left[ \frac{39634.98}{144} - \frac{158355}{4374} N_f + Kt^\sigma \left( \frac{395}{216} N_f - \frac{1533}{144} \right) \right] \quad (B17)$$

$$c_5(K, t, \sigma, T_2) = T_2 \left[ \frac{39634.98}{144} - \frac{158355}{4374} N_f + Kt^\sigma \left( \frac{395}{216} N_f - \frac{1533}{144} \right) \right] \\ + T_2^2 (3866.215 + 218339.68 Kt^\sigma) \quad (\text{B18})$$

$$c_7(K, t, \sigma, T_2) = -12 + \frac{16}{3Kt^\sigma} - T_2 \left( \frac{989}{108} N_f + 4\pi^2 + \frac{2557}{32} - \frac{20}{Kt^\sigma} \right) \\ + T_2 \left[ \frac{1}{Kt^\sigma} \left( \frac{5104}{81} + 4\pi^2 + \frac{208}{27} N_f \right) \right] \\ + T_2^2 \left( -139453.698 - \frac{22436.898}{Kt^\sigma} \right) \quad (\text{B19})$$

$$d_2(K, t, \sigma, T_1) = T_1 \left( \frac{80}{27} N_f + \frac{20}{3} Kt^\sigma \right) \quad (\text{B20})$$

$$d_4(K, t, \sigma, T_1) = \frac{1}{KT^\sigma} \left( KT^\sigma + \frac{8}{3} \right) + T_1 \left[ \frac{3843}{48} - \frac{61}{9} N_f - 3\pi^2 \right. \\ \left. + \frac{1}{KT^\sigma} \left( \frac{517}{18} - \frac{4\pi^2}{3} - \frac{80}{27} N_f \right) \right] \quad (\text{B21})$$

$$d_6(K, t, \sigma, T_2) = T_2 \left( \frac{80}{27} N_f + \frac{20}{3} Kt^\sigma \right) \\ + T_2^2 (-1908740 + 4825.06 \times 8Kt^\sigma) \quad (\text{B22})$$

$$d_8(K, t, \sigma, T_2) = \frac{1}{KT^\sigma} \left( KT^\sigma + \frac{8}{3} \right) + T_2 \left[ \frac{3843}{48} - \frac{61}{9} N_f - 3\pi^2 \right. \\ \left. + \frac{1}{KT^\sigma} \left( \frac{517}{18} - \frac{4\pi^2}{3} - \frac{80}{27} N_f \right) \right] \\ + T_2^2 \left( 135983.313 - \frac{3240.75}{Kt^\sigma} \right) \quad (\text{B23})$$

$$d_3(K, t, \sigma, T_1) = T_1 \frac{45}{2} \quad (\text{B24})$$

$$d_7(K, t, \sigma, T_2) = T_2 \frac{45}{2} \quad (\text{B25})$$

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