

The Effective Impedances of Infinite Ladder Networks and Dirichlet Problem on Graphs*

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Abstract. We calculate effective impedances of infinite LC - and CL -ladder networks as limits of effective impedances of finite network approximations, using a new method, which involves precise mathematical concepts. These concepts are related to the classical differential operators on weighted graphs. As an auxiliary result, we solve a discrete boundary value Dirichlet problem on a finite ladder network.

KEY WORDS: weighted graphs, analysis on graphs, electrical network, ladder network, Feynman's ladder, effective impedance, Laplace operator.

1 Introduction

In his *Lectures on Physics* Richard Feynman firstly states the effective impedance of an infinite LC -network [1]. Unfortunately, the result was not carefully explained. The first attempt to explain it is given in [2]. Then in [3–5] there are more investigations on this topic. Moreover, in [6] the effective impedance is calculated as the limit of effective impedances of finite network approximations, assuming that each edge has an infinitely small resistance. In the same paper the effective impedance of an infinite CL -network is calculated.

In the present paper we consider impedances as functions on $s = i\omega$, where ω is a frequency of alternating current, i is the imaginary unit, i.e. we extend a function of impedance from imaginary axis to the whole complex plane (see [7]), considering a complex frequency. Then a network is considered as a complex-weighted graph (as in previous papers by the author [8, 9]), whose weights depend on a complex parameter. The relation between electrical networks and graphs is a very wide area of research, known for several decades. Together with classical investigations on relation between direct current electrical networks and random walk on graphs (see e.g. [10–13]), there is a lot of advanced

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investigations on relation between alternating current networks and graphs (see e.g. [8, 9, 14–17]). We consider Dirichlet boundary value problems (see [9]) on finite approximations of an infinite ladder. Afterwards, we investigate a convergence of the sequences of effective impedances of the finite approximations for LC - and CL - ladders in different domains of the complex plane s . We prove that the limit for these infinite ladders exists everywhere, except some segments on the imaginary axis (see Figures 5 and 9). The way to calculate the corresponding impedances is given by equations (13) and (17).

Therefore, we calculate the known effective impedances of ladder networks using a new mathematical method, involving precise mathematical definitions. Moreover, we extend the result for $s = i\omega$, $\omega > 0$, to the whole complex plane s , not restricting ourselves to steady states.

2 Definition of a Network

Let us remind mathematical definitions of an electrical network and an effective impedance for finite and infinite networks (see [8]).

Let (V, E) be a locally finite connected graph, where V is a set of vertices, $|V| \geq 2$, and E is a set of (unoriented) edges.

Assume that each edge xy , where $x, y \in V$, is equipped with a resistance R_{xy} , inductance L_{xy} , and capacitance C_{xy} , where $R_{xy}, L_{xy} \in [0, +\infty)$ and $C_{xy} \in (0, +\infty]$, which correspond to the physical resistor, inductor (coil), and capacitor (see e.g. [18, 19]). It will be convenient to use the inverse capacity

$$D_{xy} = \frac{1}{C_{xy}} \in [0, +\infty).$$

We always assume that for any edge $xy \in E$

$$R_{xy} + L_{xy} + D_{xy} > 0 \text{ and } R_{xy}, L_{xy}, D_{xy} \geq 0.$$

The *impedance* of the edge xy is defined as the following function of a complex parameter s :

$$z_{xy}^{(s)} = R_{xy} + L_{xy}s + \frac{D_{xy}}{s}.$$

The case $s = i\omega$, $\omega > 0$ corresponds to a steady-state AC-circuit with frequency ω . We will allow s to take arbitrary values in $\mathbb{C} \setminus \{0\}$ (cf. [7–9]).

In fact, it will be more convenient to work with the *admittance* $\rho_{xy}^{(s)}$:

$$\rho_{xy}^{(s)} := \frac{1}{z_{xy}^{(s)}} = \frac{s}{L_{xy}s^2 + R_{xy}s + D_{xy}}. \quad (1)$$

Infinite networks and Dirichlet problem

Define the *physical Laplacian* Δ_ρ as an operator on functions $f : V \rightarrow \mathbb{C}$ as follows

$$\Delta_\rho f(x) = \sum_{y \in V: y \sim x} (f(y) - f(x))\rho_{xy}^{(s)}, \quad (2)$$

where $x \sim y$ means that $xy \in E$. For convenience let us extend $\rho_{xy}^{(s)}$ to all pairs $x, y \in V$ by setting $\rho_{xy}^{(s)} \equiv 0$ if $x \not\sim y$. Then the summation in (2) can be extended to all $y \in V$.

Let us fix a vertex $a_0 \in V$ and a non-empty subset $B \in V$ such that $a_0 \notin B$. Set $B_0 = B \cup \{a_0\}$. The physical meaning of a_0 and B is as follows: at the point a_0 we keep a potential 1, while the set B represents the ground. We refer to the structure $\Gamma = (V, \rho, a_0, B)$ as an (*electrical*) *network*.

On any finite network we can consider the following discrete boundary value Dirichlet problem

$$\begin{cases} \Delta_\rho v^{(s)}(x) = 0 & \text{for } x \in V \setminus B_0, \\ v^{(s)}(x) = 0 & \text{for } x \in B, \\ v^{(s)}(a_0) = 1, \end{cases} \quad (3)$$

where $v^{(s)} : V \rightarrow \mathbb{C}$ is unknown voltages at the nodes. This system follows from Ohm's and Kirchhoff's complex laws. The physical voltage at a vertex x at time t for a steady state is then $v(x)e^{i\omega t}$ for $s = i\omega$, where $\omega > 0$ is a frequency of an alternating voltage (see e.g. [18, 19]).

Let us define for any network Γ the set

$$S = \{s \in \mathbb{C} \setminus \{0\} \mid z_{xy}^{(s)} \neq 0 \text{ for any edge of } \Gamma\}.$$

Further we will consider the Dirichlet problem (3) on a given network exclusively for $s \in S$.

If $v^{(s)}(x)$ is a solution of (3) then the total current through a_0 is equal to

$$\sum_{x \in V} (1 - v^{(s)}(x))\rho_{xa_0}^{(s)},$$

which motivates the following definition (cf. [8]).

Definition 1. For any $s \in S$, the *effective admittance* of the network Γ is defined by

$$\mathcal{P}(s) = \sum_{x \in V} (1 - v^{(s)}(x))\rho_{xa_0}^{(s)}, \quad (4)$$

where v is a solution of the Dirichlet problem (3). The *effective impedance* of Γ is defined by

$$Z(s) = \frac{1}{\mathcal{P}(s)} = \frac{1}{\sum_{x \in V} (1 - v^{(s)}(x))\rho_{xa_0}^{(s)}}.$$

If the Dirichlet problem (3) has no solution for some s , then we set $\mathcal{P}(s) = \infty$ and $Z(s) = 0$.

Note that $Z(s)$ and $\mathcal{P}(s)$ take values in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In [8] is shown that in the case when (3) has multiple solutions, the values of $Z(s)$ and $\mathcal{P}(s)$ are independent of the choice of the solution v .

The defined effective impedance is a natural generalization of effective resistance for weighted graphs (see e.g. [10, 12, 13, 16]). We would like to point out, that in the case of resistances there is a strong connection between electrical networks and random walk on graphs (see e.g. [10, 13, 17]), but in the case of other passive elements we lost this connection.

The approach to an effective impedance for infinite networks, used in the following, is described in [8]. Moreover, it is a generalization of the classical approach to infinite weighted graphs (see e.g. [11, 13, 20, 21]). The idea is to consider finite network approximations.

Let $\Gamma = (V, \rho, a_0, B)$ be an infinite network. Let us consider the sequence of finite complex-weighted graphs $(V_n, \rho|_{V_n})$, where $V_n = \{x \in V \mid \text{dist}(a_0, x) \leq n\}$, $n \in \mathbb{N}$. We denote by

$$\partial V_n = \{x \in V \mid \text{dist}(a_0, x) = n\}$$

the *boundary* of the graph $(V_n, \rho|_{V_n})$. Note that $V_{n+1} = \partial V_{n+1} \cup V_n$. Let us denote $B_n = B \cap V_n$.

Then

$$\Gamma_n = (V_n, \rho|_{V_n}, a_0, B_n \cup \partial V_n), n \in \mathbb{N}$$

is a *sequence of finite networks exhausting the infinite network* Γ .

Let $\mathcal{P}_n(s)$ be the effective impedance of Γ_n .

Definition 2. Define the *effective admittance* of Γ as

$$\mathcal{P}(s) = \lim_{n \rightarrow \infty} \mathcal{P}_n(s)$$

for those $s \in \mathbb{C} \setminus \{0\}$ where the limit exists.

It is reasonable to define the effective impedance of infinite network as $Z(s) = 1/\mathcal{P}(s)$. It takes values in $\overline{\mathbb{C}}$.

Note that from the proof of Theorem 6 in [8] follows, that the effective impedance of infinite network on the positive half-plane s does not depend on the choice of an exhausting sequence of finite networks (since a limit of any sequence of effective admittances of finite networks is a holomorphic function on the positive half-plane s , which coincides with effective admittance (2) for all positive numbers).

Let us point out that in this model it is assumed that any network has potential zero at the infinity. Therefore an approach to the ladder network here is not completely the same as in [6], but it will give the same result. In [6] the ladder network is considered more like a fractal (compare to [14, 22]).

3 Finite Ladder Network

In this section we calculate the effective admittance of *any* finite ladder network. The calculation follows the same outline as in [8], but we present it here for completeness.

Consider the finite graph (V, E) , where

$$V = \{0, 1, 2, 3, 4, \dots, (2n - 3), (2n - 2)\} \cup \{2n\}$$

and E is given by $(2k - 2) \sim 2k$, $k = \overline{1, n}$ and $(2k - 1) \sim 2k$ for $k = \overline{1, (n - 1)}$. Let us consider a network as on Figure 1, where 0 is an input node, nodes $1, 3, \dots, (2n - 3)$ and $2n$ are grounded.

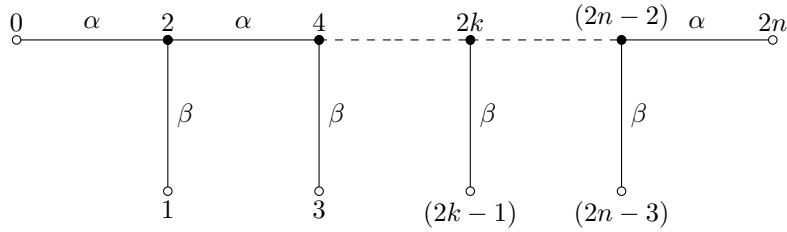


Figure 1. Finite ladder network

That is, let admittances of the edges $(2k - 2) \sim 2k$ be $\alpha = \alpha^{(s)}$ and admittances of the edges $2k - 1 \sim 2k$ be $\beta^{(s)}$. Set also $a_0 = 0$, while

$$B = \{1, 3, \dots, 2n - 3\} \cup \{2n\}.$$

We will refer to such a network as a *finite $\alpha\beta$ -network* and denote it by $\Gamma_n^{\alpha\beta}$. The Dirichlet problem (3) for this network is as follows:

$$\begin{cases} v(2k - 2) + \mu v(2k - 1) + v(2k + 2) - (2 + \mu)v(2k) = 0, & k = \overline{1, n - 1}, \\ v(0) = 1, \\ v(2k - 1) = 0, & k = \overline{1, n - 1}, \\ v(2n) = 0, \end{cases} \quad (5)$$

where $\mu = \frac{\beta}{\alpha}$.

Substituting the equations from the third line of (5) to the first line and denoting $v_k = v(2k)$, we obtain the following recurrence relation for v_k :

$$v_{k+1} - (2 + \mu)v_k + v_{k-1} = 0. \quad (6)$$

The characteristic polynomial of (6) is

$$\psi^2 - (2 + \mu)\psi + 1 = 0. \quad (7)$$

By the definition of a network $\mu \neq 0$ (i.e. $\mu^{(s)} \neq 0$ for $s \in S$). If $\mu \neq -4$, then the equation (7) has two different complex roots ψ_1, ψ_2 and its solution is

$$v_k = c_1 \psi_1^k + c_2 \psi_2^k, \quad (8)$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants.

We use the second and fourth equations of (5) as boundary conditions for this recurrence equation. Substituting (8) in the boundary conditions we obtain the following equations for the constants:

$$\begin{cases} c_1 + c_2 = 1, \\ c_1 \psi_1^n + c_2 \psi_2^n = 0. \end{cases}$$

Therefore,

$$\begin{cases} c_1 = \frac{1}{1 - \psi_1^{2n}} = \frac{-\psi_2^{2n}}{1 - \psi_2^{2n}}, \\ c_2 = \frac{1}{1 - \psi_2^{2n}} = \frac{-\psi_1^{2n}}{1 - \psi_1^{2n}}, \end{cases}$$

since $\psi_1 \psi_2 = 1$ by (7).

Now we can calculate the effective admittance of $\Gamma_n^{\alpha\beta}$ in the case $\mu \neq -4$:

$$\begin{aligned} \mathcal{P}_n^{\alpha\beta} &= \alpha (1 - v(2)) = \alpha (1 - v_1) = \alpha (1 - c_1 \psi_1 - c_2 \psi_2) \\ &= \frac{\alpha (\psi_1^{2n-1} + 1) (\psi_1 - 1)}{(\psi_1^{2n} - 1)} = \frac{\alpha (\psi_2^{2n-1} + 1) (\psi_2 - 1)}{(\psi_2^{2n} - 1)}. \end{aligned} \quad (9)$$

By definition, an effective admittance $\mathcal{P}_n^{\alpha\beta}$ is a rational function of α and β . Indeed, using formula for the roots of equation (7) and binomial expansion, it can be written as a rational function of α and β , without usage of ψ_1, ψ_2 :

$$\begin{aligned} \mathcal{P}_n^{\alpha\beta} &= \alpha (1 - c_1 \psi_1 - c_2 \psi_2) = \alpha \left(1 - \frac{\psi_1}{1 - \psi_1^{2n}} - \frac{\psi_2}{1 - \psi_2^{2n}} \right) \\ &= \alpha \left(1 - \frac{\psi_1 (1 - \psi_2^{2n}) + \psi_2 (1 - \psi_1^{2n})}{(1 - \psi_1^{2n})(1 - \psi_2^{2n})} \right) \\ &= \alpha \left(1 - \frac{\psi_1 + \psi_2 - (\psi_2^{2n-1} + \psi_1^{2n-1})}{2 - (\psi_1^{2n} + \psi_2^{2n})} \right) \\ &= \alpha \left(1 - \frac{2 + \frac{\beta}{\alpha} - 2 \sum_{k=0}^{n-1} \binom{2n-1}{2k} \left(1 + \frac{\beta}{2\alpha} \right)^{2n-2k-1} \left(\frac{\beta}{\alpha} + \left(\frac{\beta}{2\alpha} \right)^2 \right)^k}{2 - 2 \sum_{k=0}^n \binom{2n}{2k} \left(1 + \frac{\beta}{2\alpha} \right)^{2n-2k} \left(\frac{\beta}{\alpha} + \left(\frac{\beta}{2\alpha} \right)^2 \right)^k} \right), \end{aligned}$$

since $\psi_1 + \psi_2 = 2 + \mu = 2 + \frac{\beta}{\alpha}$ by (7).

Let us now consider the case $\mu = -4$. Then the solution of the recurrence relation (6) is

$$v_k = c_1(-1)^k + c_2k(-1)^k,$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants. And using boundary conditions for the recurrence relation, we obtain

$$\begin{cases} c_1 = 1 \\ c_2 = -\frac{1}{n}. \end{cases}$$

Then the effective admittance is

$$\mathcal{P}_n = \alpha(1 - v(2)) = \alpha(1 - v_1) = \alpha\left(1 - \left(-1 + \frac{1}{n}\right)\right) = \frac{\alpha(2n - 1)}{n}.$$

Therefore, for a finite $\alpha\beta$ -network we have for any $s \in S$

$$\mathcal{P}_n = \begin{cases} \frac{\alpha(2n - 1)}{n}, & \text{if } \frac{\beta}{\alpha} = -4, \\ \mathcal{P}_n^{\alpha\beta}, & \text{otherwise.} \end{cases} \quad (10)$$

4 Feynman's Ladder (LC -Network) with Zero at Infinity

In this section we calculate the effective admittance of an infinite LC -network, using its finite approximations and their admittances, calculated in the previous section. Consider the infinite ladder network (V, E) , where

$$V = \{0, 1, 2, 3, 4, \dots\}$$

and E is given by $(2k - 2) \sim 2k$ and $(2k - 1) \sim 2k$ for $k = \overline{1, \infty}$ (see Figure 2)

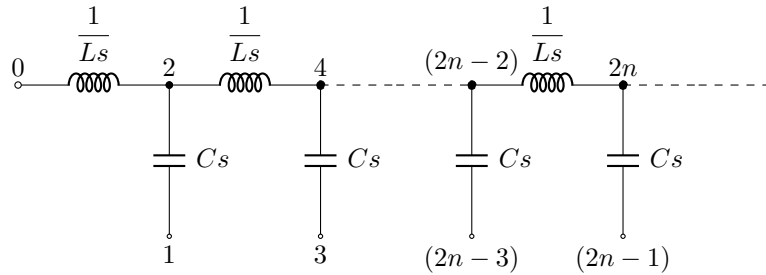


Figure 2. Feynman's ladder with zero at infinity

Let the admittance of the edges $(2k - 2) \sim 2k$ be $\frac{1}{Ls}$ and admittance of the edges $2k - 1 \sim 2k$ be Cs , where $L > 0$ and $C > 0$. Set also $a_0 = 0$, while $B = \{1, 3, \dots\}$.

This network is the Feynman's ladder network (see [1]) and we assume that it has a ground at infinity.

In this section we analyse the behaviour of the sequence of the effective admittances for the exhausted networks in the whole complex plane s , and compare our calculations with the result stated by Richard Feynman in [1] and Sung Hyun Yoon in [6] for $s = i\omega, \omega > 0$.

This network can be exhausted by finite $\alpha\beta$ -networks (with $\alpha = \frac{1}{Ls}, \beta = Cs$), whose effective admittances by (9) and (10) are

$$\mathcal{P}_n(s) = \begin{cases} \frac{2n-1}{Lsn}, & \text{if } \mu = -4, \\ \text{not defined,} & \text{if } s = 0, \\ \mathcal{P}_n^{\alpha\beta}(s) = \frac{(\psi_1^{2n-1} + 1)(\psi_1 - 1)}{Ls(\psi_1^{2n} - 1)}, & \text{otherwise,} \end{cases}$$

where $\mu = LCs^2$ and ψ_1 is any root of the equation

$$\psi^2 - (2 + LCs^2)\psi + 1 = 0. \quad (11)$$

Let us analyse the sequence $\{\mathcal{P}_n^{\alpha\beta}(s)\}, n \rightarrow \infty$.

Since ψ_1 is any root of the quadratic equation (11), we can assume, without loss of generality, that $|\psi_1| \leq 1 \leq |\psi_2|$. Note that then $\{\mathcal{P}_n^{\alpha\beta}(s)\}$ has limit if and only if $|\psi_1| < 1$. This limit is equal to $(1 - \psi_1)/(Ls)$. Now we will reformulate the condition on existence of the limit in terms of s . Firstly, we will prove the following claim.

Claim 3. Let $s \in \mathbb{C} \setminus \{0\}, s^2 \neq -\frac{4}{LC}$. Then the condition $|\psi_1| = |\psi_2| = 1$ occurs if and only if $s^2 \in \left(-\frac{4}{LC}, 0\right)$.

Proof. “ \Rightarrow ” Let $|\psi_1| = 1$. Then $|\psi_2| = 1$, since $\psi_1\psi_2 = 1$ (see (11)). Further, since $LCs^2 \neq 0$ and $LCs^2 \neq -4$, it follows from (11) and $|\psi_1| = |\psi_2| = 1$ that $\frac{\psi_1}{\psi_2}, \frac{\psi_2}{\psi_1} \notin \mathbb{R}$ (since $\psi_1, \psi_2 \neq \pm 1$). And, using $\psi_1\overline{\psi_1} = |\psi_1|^2 = 1$, we have $\psi_2 = \overline{\psi_1}$. Moreover, by (11) we conclude that $2 + LCs^2 = \psi_1 + \psi_2 = 2\operatorname{Re}\psi_1 \in \mathbb{R}$, i.e. $LCs^2 \in \mathbb{R}$. Then $\psi_1, \psi_2 \notin \mathbb{R}$ means that the discriminant of (11)

$$(2 + LCs^2)^2 - 4 = 4LCs^2 + (LCs^2)^2$$

is negative, i.e. $LCs^2 \in (-4, 0)$. Therefore,

$$s^2 \in \left(-\frac{4}{LC}, 0\right)$$

which was to be proved.

“ \Leftarrow ” Let $s^2 \in \left(-\frac{4}{LC}, 0\right)$. Then the discriminant of (11) is negative and

$$\begin{aligned} |\psi_{1,2}|^2 &= \left| 1 + \frac{LCs^2}{2} \pm i\sqrt{-LCs^2 - \left(\frac{LCs^2}{2}\right)^2} \right|^2 \\ &= \left(1 + \frac{LCs^2}{2}\right)^2 - LCs^2 - \left(\frac{LCs^2}{2}\right)^2 = 1. \end{aligned}$$

□

Therefore, for any $s \in \mathbb{C} \setminus \left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}}\right]$ we can write

$$\psi_1(s) = 1 + \frac{LCs^2}{2} + s\xi(s), \quad (12)$$

where $\xi(s)$ is the square root of $LC + \frac{L^2C^2s^2}{4}$, such that $|\psi_1(s)| < 1$.

Let us denote $\gamma = LC + \frac{L^2C^2s^2}{4}$. Let us consider $\gamma \in \mathbb{C} \setminus (-i\infty, 0)$, i.e.

$\gamma = re^{i\phi}$, $\phi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ in polar coordinates. Then there are exactly two continuous functions ξ_1, ξ_2 , which give square roots of γ :

$$\xi_1(\gamma) = \sqrt{r}e^{i\frac{\phi}{2}}, \xi_2(\gamma) = -\xi_1(\gamma)$$

(see Figure 3).

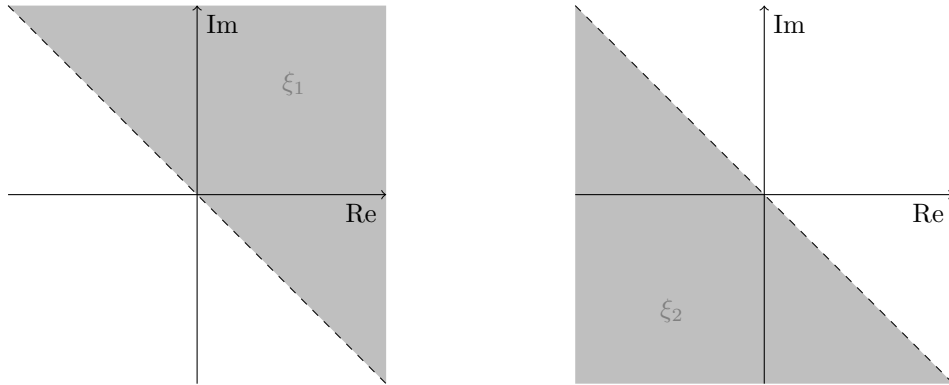


Figure 3. The images of ξ_1 and ξ_2

Since

$$\begin{aligned}\gamma &= LC + \frac{L^2 C^2 s^2}{4} = LC + \frac{L^2 C^2 (\operatorname{Re} s + i \operatorname{Im} s)^2}{4} \\ &= LC + \frac{L^2 C^2 (\operatorname{Re} s)^2 - L^2 C^2 (\operatorname{Im} s)^2}{4} + i \frac{L^2 C^2 (\operatorname{Re} s)(\operatorname{Im} s)}{2},\end{aligned}$$

$\xi_1(\gamma(s))$ and $\xi_2(\gamma(s))$ are defined for all $s \in \mathbb{C} \setminus \bar{S}$, where

$$\bar{S} = \left\{ 1 + \frac{LC (\operatorname{Re} s)^2 - LC (\operatorname{Im} s)^2}{4} = 0 \text{ and } (\operatorname{Re} s)(\operatorname{Im} s) < 0 \right\},$$

see Figure 4.

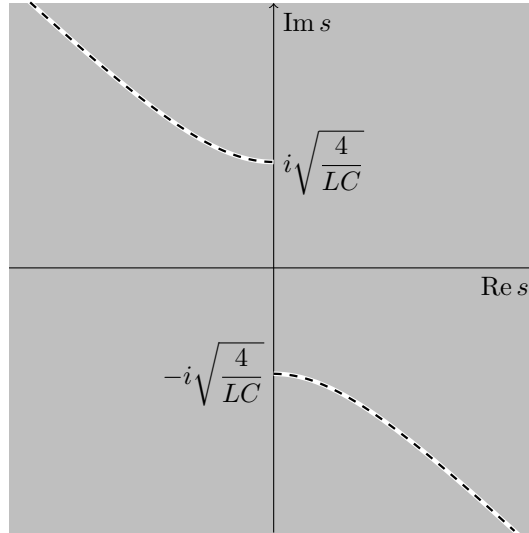


Figure 4. The domain of $\xi_{1,2}(\gamma(s))$

Since the functions

$$\left| 1 + \frac{LC s^2}{2} + s \xi_1(\gamma(s)) \right| \text{ and } \left| 1 + \frac{LC s^2}{2} + s \xi_2(\gamma(s)) \right|$$

are continuous on s , the choice of the function $\xi_1(s)$ or $\xi_2(s)$ in (12), can not change inside the domains

$$\Omega_1 = \left\{ \operatorname{Re}^2 s - \operatorname{Im}^2 s > -\sqrt{\frac{4}{LC}}, \operatorname{Re} s > 0 \right\} \cup \left\{ \operatorname{Re}^2 s - \operatorname{Im}^2 s < -\sqrt{\frac{4}{LC}}, \operatorname{Im} s > 0 \right\}$$

and

$$\Omega_2 = \left\{ \operatorname{Re}^2 s - \operatorname{Im}^2 s > -\sqrt{\frac{4}{LC}}, \operatorname{Re} s < 0 \right\} \cup \left\{ \operatorname{Re}^2 s - \operatorname{Im}^2 s < -\sqrt{\frac{4}{LC}}, \operatorname{Im} s < 0 \right\}$$

(see Figure 5).

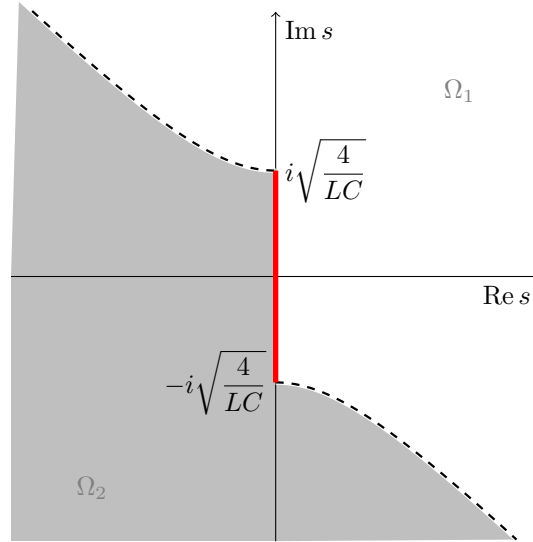


Figure 5. The domains Ω_1 and Ω_2

Taking $s = 2 \in \Omega_1$ we have

$$\begin{aligned} \left| 1 + \frac{LCs^2}{2} + s\xi_1(\gamma(s)) \right| &= \left| 1 + 2LC + 2\xi_1(LC + L^2C^2) \right| \\ &= \left| 1 + 2LC + 2\sqrt{LC + L^2C^2} \right| > 1. \end{aligned}$$

Therefore,

$$\psi_1(s) = 1 + \frac{LCs^2}{2} + s\xi_2(\gamma(s)), s \in \Omega_1.$$

In the same way, taking $s = -2 \in \Omega_2$ we obtain

$$\begin{aligned} \left| 1 + \frac{LCs^2}{2} + s\xi_1(\gamma(s)) \right| &= \left| 1 + 2LC - 2\xi_1(LC + L^2C^2) \right| \\ &= \left| 1 + 2LC - 2\sqrt{LC + L^2C^2} \right| < 1. \end{aligned}$$

Therefore,

$$\psi_1(s) = 1 + \frac{LCs^2}{2} + s\xi_1(\gamma(s)), s \in \Omega_2.$$

Therefore, we can calculate the effective admittance of the infinite network for any $s \in \mathbb{C} \setminus \left(\overline{S} \cup \left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}} \right] \right)$ as

$$\mathcal{P}(s) = \frac{1 - \psi_1}{Ls} = \begin{cases} -\frac{Cs}{2} - \frac{\xi_2(\gamma(s))}{L}, & s \in \Omega_1, \\ -\frac{Cs}{2} - \frac{\xi_1(\gamma(s))}{L}, & s \in \Omega_2. \end{cases} \quad (13)$$

The effective admittance for the $s \in \overline{S}$ we can calculate, considering another cut of the plane γ . Moreover, since the cut $(-i\infty, 0)$ has been chosen arbitrary, the limits of the effective admittances from the both sides of the curves \overline{S} will coincide and give the required quantity.

Therefore, we can calculate the effective admittance of the infinite network for any $s \in \mathbb{C} \setminus \left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}} \right]$.

A natural question is whether the right and left limits of the effective admittance (or of the $\xi_1(s), \xi_2(s)$) at the segment $\left[-i\sqrt{\frac{4}{LC}}, i\sqrt{\frac{4}{LC}} \right]$ coincide (see Figure 6). The answer is negative.

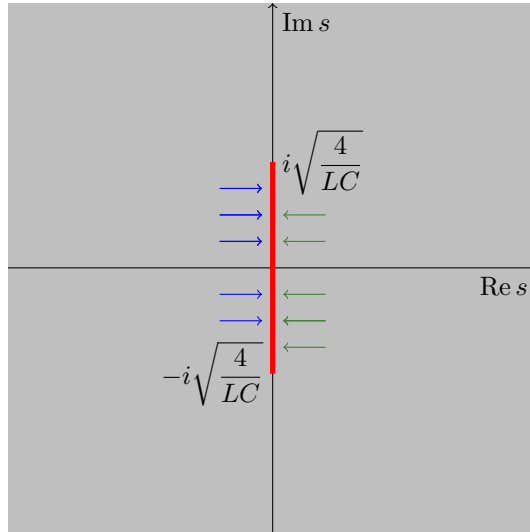


Figure 6. The limits of $\xi_1(s)$ and $\xi_2(s)$

Indeed, let $s = \epsilon + i\omega$, $1 \gg \epsilon > 0$, $\omega \in \left(-\sqrt{\frac{4}{LC}}, \sqrt{\frac{4}{LC}} \right)$, $s \in \Omega_1$. Then

$$\gamma(\epsilon + i\omega) = LC + \frac{L^2C^2}{4}(\epsilon^2 - \omega^2) + i\frac{L^2C^2}{2}\epsilon\omega$$

and

$$\begin{aligned}\lim_{s \rightarrow +i\omega} \mathcal{P}(s) &= \lim_{\epsilon \rightarrow 0} \left(-\frac{C(\epsilon + i\omega)}{2} - \frac{\xi_2(\gamma(\epsilon + i\omega))}{L} \right) \\ &= -\frac{Ci\omega}{2} + \sqrt{\frac{C}{L} - \frac{C^2\omega^2}{4}},\end{aligned}\quad (14)$$

since $\text{Re } \gamma > 0$, $|\text{Im } \gamma| \ll 1$ provides $\text{Re } \xi_2 < 0$.

Let $s = -\epsilon + i\omega$, $1 \gg \epsilon > 0$, $\omega \in \left(-\sqrt{\frac{4}{LC}}, \sqrt{\frac{4}{LC}} \right)$, $s \in \Omega_2$. Then

$$\gamma(-\epsilon + i\omega) = LC + \frac{L^2C^2}{4} (\epsilon^2 - \omega^2) - i\frac{L^2C^2}{2}\epsilon\omega$$

and

$$\begin{aligned}\lim_{s \rightarrow -i\omega} \mathcal{P}(s) &= \lim_{\epsilon \rightarrow 0} \left(-\frac{C(-\epsilon + i\omega)}{2} - \frac{\xi_1(\gamma(-\epsilon + i\omega))}{L} \right) \\ &= -\frac{Ci\omega}{2} - \sqrt{\frac{C}{L} - \frac{C^2\omega^2}{4}},\end{aligned}$$

since $\text{Re } \gamma > 0$, $|\text{Im } \gamma| \ll 1$ provides $\text{Re } \xi_1 > 0$.

The limit (14) coincides with the one, stated by R. Feynman in [1] and S.H. Yoon in [6]. Indeed, by [1, p. 22-13] we have the effective admittance

$$\begin{aligned}\mathcal{P} &= \frac{1}{Z} = \frac{1}{i\omega L/2 + \sqrt{(L/C) - (\omega^2 L^2/4)}} = \frac{i\omega L/2 - \sqrt{(L/C) - (\omega^2 L^2/4)}}{(i\omega L/2)^2 - ((L/C) - (\omega^2 L^2/4))} \\ &= -\frac{C}{L} \left(\frac{i\omega L}{2} - \sqrt{(L/C) - (\omega^2 L^2/4)} \right) = -\frac{Ci\omega}{2} + \sqrt{\frac{C}{L} - \frac{C^2\omega^2}{4}}.\end{aligned}$$

This corresponds to the ideas in [14] and in [6] to calculate the effective impedance as right half-plane limit. Physically it makes sense, since the real resistance in any part of a physical network is always greater than zero. Note that we have made a rescaling in a different way than in the above-mentioned papers, although it leads to the same result. Namely, we add positive real component to $i\omega$, but not to the whole edge.

5 CL-Network with Zero at Infinity

In this section we calculate the effective impedance of infinite CL - network, using finite network approximation, described in Section 3. Consider the infinite ladder network (V, E) , where

$$V = \{0, 1, 2, 3, 4, \dots\}$$

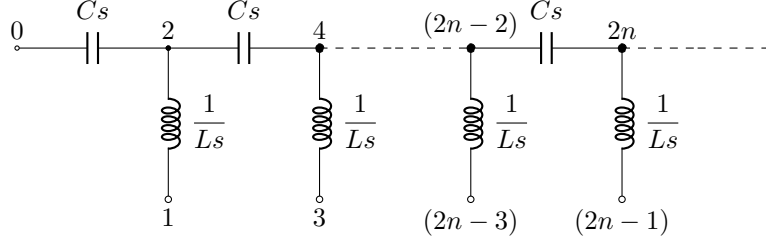


Figure 7. CL -ladder with zero at infinity

and E is given by $(2k - 2) \sim 2k$ and $(2k - 1) \sim 2k$ for $k = \overline{1, \infty}$ (see Figure 7)

Let the admittance of the edges $(2k - 2) \sim 2k$ be Cs and admittance of the edges $2k - 1 \sim 2k$ be $\frac{1}{Ls}$, where $C, L > 0$. Set also $a_0 = 0$, while $B = \{1, 3, \dots\}$.

This network is a CL -ladder network (see [6]). Note that we assume it has a ground at infinity.

We will calculate an effective admittance of the CL -ladder in the same way, as we did for Feynman's ladder in Section 4, and compare our calculations to the result stated by Sung Hyun Yoon in [6] for $s = i\omega$, $\omega > 0$.

The CL -network can be exhausted by finite $\alpha\beta$ -networks (with $\alpha = Cs$, $\beta = \frac{1}{Ls}$), whose effective admittances by (9) and (10) are

$$\mathcal{P}_n(s) = \begin{cases} \frac{Cs(2n-1)}{n}, & \text{if } s = \pm \frac{i}{2\sqrt{CL}}, \\ \text{not defined,} & \text{if } s = 0, \\ \mathcal{P}_n^{\alpha\beta}(s) = \frac{Cs(\psi_1^{2n-1} + 1)(\psi_1 - 1)}{(\psi_1^{2n} - 1)}, & \text{otherwise,} \end{cases}$$

where ψ_1 is any root of the equation

$$\psi^2 - \left(2 + \frac{1}{CLs^2}\right)\psi + 1 = 0. \quad (15)$$

Firstly, let us point out that

$$\lim_{n \rightarrow \infty} \mathcal{P}_n\left(\pm \frac{i}{2\sqrt{CL}}\right) = \lim_{n \rightarrow \infty} \pm \frac{Ci}{2\sqrt{CL}} \frac{(2n-1)}{n} = \pm i\sqrt{\frac{C}{L}}.$$

Let us analyse the sequence $\{\mathcal{P}_n^{\alpha\beta}(s)\}$, $n \rightarrow \infty$.

Since ψ_1 is any root of the quadratic equation (15), we can assume, without loss of generality, that $|\psi_1| \leq 1 \leq |\psi_2|$. Note that then $\{\mathcal{P}_n^{\alpha\beta}(s)\}$ has limit if and only if $|\psi_1| < 1$. This limit is equal to $Cs(1 - \psi_1)$.

Claim 4. Let $s \in \mathbb{C} \setminus \{0\}$, $s^2 \neq -\frac{1}{4CL}$. Then the condition $|\psi_1| = |\psi_2| = 1$ occurs if and only if $s^2 \in \left(-\infty, -\frac{1}{4CL}\right)$.

Proof. The proof follows the same outline as the proof of Claim 3. □

Therefore, for any $s \in \mathbb{C} \setminus \left(\left(-i\infty, -\frac{i}{2\sqrt{CL}}\right] \cup \left[\frac{i}{2\sqrt{CL}}, +i\infty\right)\right)$ we can write

$$\psi_1(s) = 1 + \frac{1}{2CLs^2} + \frac{1}{2CLs^2}\xi(\gamma(s)), \quad (16)$$

where $\xi(s)$ is the square root of $4s^2CL + 1$, such that $|\psi_1(s)| < 1$.

Let us denote $\gamma = 4CLs^2 + 1$. Let us consider $\gamma \in \mathbb{C} \setminus (-i\infty, 0)$, i.e. $\gamma = re^{i\phi}$, $\phi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ in polar coordinates (the same cut of the complex plane as in Section 4). Then there are two continuous functions ξ_1, ξ_2 , which give square roots of γ (see Figure 3):

$$\xi_1(\gamma) = \sqrt{r}e^{i\frac{\phi}{2}}, \xi_2(\gamma) = -\xi_1(\gamma).$$

Since

$$\gamma = 4s^2CL + 1 = 4CL(\operatorname{Re} s)^2 - 4CL(\operatorname{Im} s)^2 + 1 + i8CL(\operatorname{Re} s)(\operatorname{Im} s)$$

$\xi_1(\gamma(s))$ and $\xi_2(\gamma(s))$ are defined for all $s \in \mathbb{C} \setminus \bar{S}$, where

$$\bar{S} = \left\{4CL(\operatorname{Re} s)^2 - 4CL(\operatorname{Im} s)^2 + 1 = 0 \text{ and } (\operatorname{Re} s)(\operatorname{Im} s) < 0\right\},$$

see Figure 8.

Since the functions

$$\left|1 + \frac{1}{2CLs^2} + \frac{1}{2CLs^2}\xi_1(\gamma(s))\right| \text{ and } \left|1 + \frac{1}{2CLs^2} + \frac{1}{2CLs^2}\xi_2(\gamma(s))\right|$$

are continuous on s , the choice of the function $\xi_1(\gamma(s))$ or $\xi_2(\gamma(s))$ in (16), can not change inside the domains

$$\Omega_1 = \left\{\operatorname{Re}^2 s - \operatorname{Im}^2 s > -\frac{1}{4CL}\right\} \cup \left\{\operatorname{Re}^2 s - \operatorname{Im}^2 s < -\frac{1}{4CL}, (\operatorname{Im} s)(\operatorname{Re} s) > 0\right\},$$

$$\Omega_2 = \left\{\operatorname{Re}^2 s - \operatorname{Im}^2 s < -\frac{1}{4CL}, \operatorname{Im} s > 0, \operatorname{Re} s < 0\right\}$$

and (see Figure 9)

$$\Omega_3 = \left\{\operatorname{Re}^2 s - \operatorname{Im}^2 s < -\frac{1}{4CL}, \operatorname{Im} s < 0, \operatorname{Re} s > 0\right\}.$$

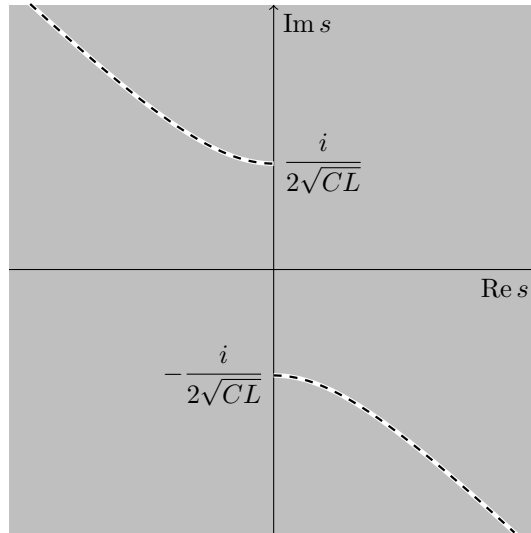


Figure 8. The domain of $\xi_{1,2}(\gamma(s))$

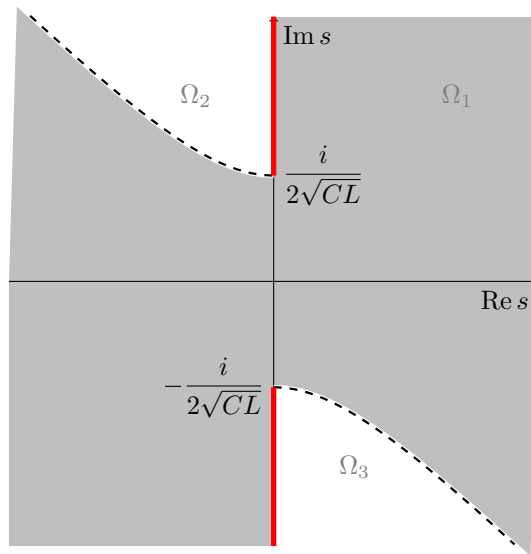


Figure 9. The domains Ω_1, Ω_2 and Ω_3 for a CL -network.

Taking $s = 2 \in \Omega_1$ we have

$$\left| 1 + \frac{1}{2CLs^2} + \frac{1}{2CLs^2} \xi_1(\gamma(s)) \right| = \left| 1 + \frac{1}{8CL} + \frac{1}{8CL} \sqrt{16CL + 1} \right| > 1.$$

Infinite networks and Dirichlet problem

Therefore,

$$\psi_1(s) = 1 + \frac{1}{2CLs^2} + \frac{\xi_2(\gamma(s))}{2CLs^2}, s \in \Omega_1.$$

For the domain Ω_2 in this case we should use a small trick, since taking square root from arbitrary complex number is not easy. This is also a reason, why a practical application of our method for arbitrary ladder network (i.e. with arbitrary $\alpha^{(s)}$ and $\beta^{(s)}$) can be complicated.

Let us take $\gamma = -3 - 4i$. Then

$$s^2 = \frac{\gamma - 1}{4CL} = -\frac{1}{CL} - \frac{i}{CL}.$$

From the other hand, $s^2 = (\operatorname{Re} s)^2 - (\operatorname{Im} s)^2 + 2(\operatorname{Re} s)(\operatorname{Im} s)$. Therefore, we have

$$(\operatorname{Re} s)^2 - (\operatorname{Im} s)^2 = -\frac{1}{CL} < -\frac{1}{4CL} \text{ and } (\operatorname{Re} s)(\operatorname{Im} s) = -\frac{1}{CL} < 0,$$

i.e. $s \in \Omega_2$ or $s \in \Omega_3$. Indeed, there exist exactly two complex numbers s_1, s_2 such that $\gamma(s_{1,2}) = -3 - 4i$. Without loss of generality we can assume, that $s_1 \in \Omega_2$. Then $s_2 \in \Omega_3$ and, due to the symmetry $\xi_1(s) = \xi_1(-s)$ and representation of $\psi_1(s)$, we have $\psi_1(s_1) = \psi_1(s_2)$. Let us calculate $\psi_1(s_1)$. Using $s_1^2 = -\frac{1}{CL} - \frac{i}{CL}$, we obtain

$$\begin{aligned} \left| 1 - \frac{1}{4} + \frac{i}{4} + \left(-\frac{1}{4} + \frac{i}{4} \right) \xi_1(-3 - 4i) \right| &= \left| 1 - \frac{1}{4} + \frac{i}{4} + \left(-\frac{1}{4} + \frac{i}{4} \right) (-1 + 2i) \right| \\ &= \left| \frac{1}{2} - i\frac{1}{2} \right| < 1. \end{aligned}$$

Therefore,

$$\psi_1(s) = 1 + \frac{1}{2CLs^2} + \frac{\xi_1(\gamma(s))}{2CLs^2}, s \in \Omega_2 \cup \Omega_3.$$

Therefore, we can calculate the effective admittance of an infinite CL -ladder for any $s \in \mathbb{C} \setminus \left(\overline{S} \cup \left(-i\infty, -\frac{i}{2\sqrt{CL}} \right) \cup \left(\frac{i}{2\sqrt{CL}}, +i\infty \right) \cup \{0\} \right)$ as

$$\mathcal{P}(s) = Cs(1 - \psi_1) = \begin{cases} -\frac{1}{2Ls} - \frac{\xi_2(\gamma(s))}{2Ls}, s \in \Omega_1, \\ -\frac{1}{2Ls} - \frac{\xi_1(\gamma(s))}{2Ls}, s \in \Omega_2 \cup \Omega_3, \\ \pm i\sqrt{\frac{C}{L}}, s = \pm \frac{i}{2\sqrt{CL}} \end{cases} \quad (17)$$

since at the point $s = 0$ the admittances of some edges are not defined.

The effective admittance for the $s \in \bar{S}$ we can calculate, considering another cut of the plane γ . Moreover, since the cut $(-\infty, 0)$ has been chosen arbitrary, the limits of the effective admittances from the both sides of the curves \bar{S} will coincide and give the required quantity.

Therefore, we can calculate the effective admittance of the infinite network for any $s \in \mathbb{C} \setminus \left((-\infty, -\frac{i}{2\sqrt{CL}}) \cup (\frac{i}{2\sqrt{CL}}, +i\infty) \cup \{0\} \right)$.

A natural question is whether the right and left limits of the effective admittance (or of the $\xi_1(s)$, $\xi_2(s)$) at the open intervals $(-\infty, -i\frac{1}{2\sqrt{CL}})$ and $(i\frac{1}{2\sqrt{CL}}, +\infty)$ coincide (see Figure 10). The answer is negative.

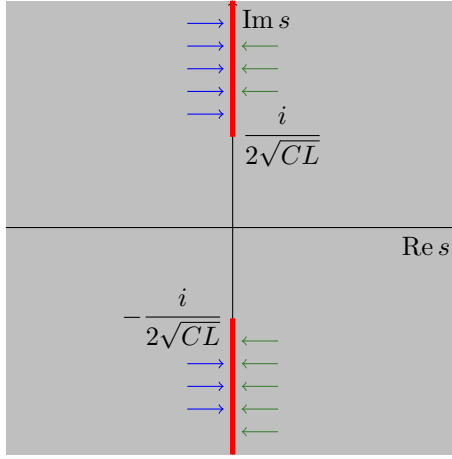


Figure 10. The limits of $\xi_1(s)$ and $\xi_2(s)$ for a CL -network

We will consider upper half case $(i\frac{1}{2\sqrt{CL}}, +i\infty)$. The other case can be calculated similarly.

Let $s = \epsilon + i\omega$, $1 \gg \epsilon > 0$, $\omega \in (\frac{1}{2\sqrt{CL}}, +\infty)$, i.e. $s \in \Omega_1$. Then

$$\gamma(\epsilon + i\omega) = 4CL(\epsilon^2 - \omega^2) + 1 + i8CL\epsilon\omega$$

and

$$\begin{aligned} \lim_{s \rightarrow +i\omega} \mathcal{P}(s) &= \lim_{\epsilon \rightarrow 0} \left(-\frac{1}{2L(\epsilon + i\omega)} - \frac{\xi_2(\gamma(\epsilon + i\omega))}{2L(\epsilon + i\omega)} \right) \\ &= -\frac{1}{2Li\omega} + \frac{i}{2Li\omega} \sqrt{4CL\omega^2 - 1} = \frac{i}{2L\omega} + \sqrt{\frac{C}{L} - \frac{1}{4L^2\omega^2}}, \quad (18) \end{aligned}$$

Infinite networks and Dirichlet problem

since $\operatorname{Re} \gamma < 0$, $|\operatorname{Im} \gamma| \ll 1$ provides $\operatorname{Im} \xi_2 < 0$.

Let $s = -\epsilon + i\omega$, $1 \gg \epsilon > 0$, $\omega \in \left(\frac{1}{2\sqrt{CL}}, +\infty\right)$, i.e. $s \in \Omega_2$. Then

$$\gamma(-\epsilon + i\omega) = 4CL(\epsilon^2 - \omega^2) + 1 - i8CL\epsilon\omega$$

and

$$\begin{aligned} \lim_{s \rightarrow -i\omega} \mathcal{P}(s) &= \lim_{\epsilon \rightarrow 0} \left(-\frac{1}{2L(-\epsilon + i\omega)} - \frac{\xi_1(\gamma(-\epsilon + i\omega))}{2L(-\epsilon + i\omega)} \right) \\ &= -\frac{1}{2Li\omega} - \frac{i}{2Li\omega} \sqrt{4CL\omega^2 - 1} = \frac{i}{2L\omega} - \sqrt{\frac{C}{L} - \frac{1}{4L^2\omega^2}}, \end{aligned}$$

since $\operatorname{Re} \gamma < 0$, $|\operatorname{Im} \gamma| \ll 1$ provides $\operatorname{Im} \xi_1 > 0$.

The limit (18) coincides with the one, stated by S.H. Yoon in [6]. Indeed, by [6, p. 286] we have the effective admittance

$$\begin{aligned} \mathcal{P} &= \frac{1}{Z} = \frac{1}{-i/(2\omega C) + \sqrt{L/C - 1/(2\omega C)^2}} \\ &= \frac{-i/(2\omega C) - \sqrt{L/C - 1/(2\omega C)^2}}{(-i/(2\omega C))^2 - (L/C - 1/(2\omega C)^2)} \\ &= -\frac{C}{L} \left(\frac{-i}{2\omega C} - \sqrt{\frac{L}{C} - \frac{1}{(2\omega C)^2}} \right) \\ &= \frac{i}{2L\omega} + \sqrt{\frac{C}{L} - \frac{1}{4L^2\omega^2}}. \end{aligned}$$

Note, that the above described method for calculation of the effective admittances for LC - and CL - networks can be used for any infinite ladder network with the admittances $\alpha^{(s)}$, $\beta^{(s)}$, although the choice of the one of two solutions of the quadratic equation

$$\psi^2 - (2 + \mu)\psi + 1 = 0. \quad (19)$$

in different domains of the complex plane s can be practically quite complicated for a given infinite ladder network. Therefore, we have presented a general method for calculation of the effective impedance of infinite ladder network and illustrated it with two physically important infinite ladder networks.

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