

A Magnetized Kantowski-Sachs Inflationary Universe in General Relativity

S.D. Katore

PG Department of Mathematics, SGB Amravati University, Amravati, India

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Abstract. In this paper the magnetized Kantowski-Sachs inflationary cosmological model in the presence of massless scalar field with a flat potential is investigated. To get an inflationary universe, we have considered a flat region in which the potential V is constant. The magnetic field is due to an electric current produced along the x -axis. Thus the magnetic field is in the yz -plane and F_{23} is the only non-vanishing component of the electromagnetic field tensor F_{ij} . To get the deterministic solutions in terms of cosmic time t , we have used negative constant deceleration parameter q proposed by Berman. Some physical and kinematical parameters of the model are also discussed.

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1 Introduction

The inflationary universe resolves some of the most outstanding problems of standard cosmology including the horizon problem, the structure problem and the massive relics problem of the observed universe. The basic idea of inflation has to do with the rate at which the universe is expanding. The idea of an early inflationary universe has been introduced by Alan Guth [1] in the context of Grand Unified theory to solve this problem of our present universe described by the standard hot big-bang cosmology. In inflationary scenario, the de-Sitter expansion is induced by vacuum energy density in general relativity. Several versions of the inflationary models have been studied by Linde [2], La and Steinhardt [3], Abbott and Wise [4]. In particular, our universe is homogeneous and isotropic to a very high degree of precision. Such an universe can be described by the well-known Friedmann–Robertson–Walker (FRW) metric. In these models, the flatness problem is well understood, but the situation is not so clear about isotropy and homogeneity to solve such problem.

However FRW models are solving the horizon, a necessary, but not sufficient condition for the solution of the homogeneous Bianchi-types even isotropy and

homogeneity problems. Bekin [5], Jensen and Stein-Schabes [6], Rothman and Ellis [7] have investigated the solution of an isotropic problem when we work with anisotropic metrics and show that they can be isotropized and inflated under general circumstances. In these models the universe undergoes a phase transition characterized by the evolution of a Higg's field ϕ towards the minimum potential $V(\phi)$. The Higg's field was initially a zero vacuum expectation value (VEV). Inflation will take place if the potential $V(\phi)$ has a flat region and the scalar field ϕ evolves slowly but the universe expands in an exponential way due to matterless scalar field. In general relativity, scalar field helps in expanding the creation of matter in the cosmological theories and can also describe the unchanged field. The scalar field is minimally coupled to the gravitational field. In particular, the self-interacting scalar fields play a central role in the study of inflationary cosmological model.

Wald [8], Burd and Barrow [9], Barrow [10], Ellis and Madsen [11] and Heusler [12] have studied different aspect of scalar fields in the evolution of the universe and FRW models. Bhattacharjee and Baruah [13], Bali and Jain [14] and Rahaman *et al.* [15] have studied the role of self-interacting scalar fields in inflationary cosmology. Recently, Reddy *et al.* [16], Reddy and Naidu [17], Katore and Rane [18] have studied inflationary universe models in four and five dimensions in general relativity.

The present day magnitude of the magnetic energy is very small in comparison with the estimated matter density. It might not have been negligible during early stage of the evolution of the universe. A cosmological model which contains a global magnetic field is necessarily anisotropic since the magnetic field vector specifies a preferred spatial direction. The primordial magnetic field of cosmological origin has been speculated by Asseo and Sol [19]. FRW models are approximately valid as present day magnetic field strength is very small. The break down of isotropy is also due to the magnetic field. In this paper, using the concept of Higg's field with flat potential, Kantowski–Sachs inflationary cosmological model has been obtained in the presence of an electromagnetic field. To get inflationary solution we have assumed a massless scalar field with the potential $V(\phi)$ that has flat potential. We also assumed a negative constant deceleration parameter proposed by Bermann [20]. Magnetized Kantowski–Sachs inflationary model has astrophysical interest since the cosmological models play a vital role in the structure formation of the universe.

2 Metric and Field Equations

The homogeneous and anisotropic universe is described by Kantowski-Sachs space-time in the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where the metric potentials A and B are functions of cosmic time t alone.

In this case of gravity minimally coupled to a scalar field $V(\phi)$, the Lagrangian L is given by

$$L = \int [R - \frac{1}{2}g^{ij} \phi_{,i}\phi_{,j} - V(\phi)]\sqrt{-g}d^4x \quad (2)$$

which on variation of L with respect to dynamical fields lead to Einstein's field equations

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij}. \quad (3)$$

The energy momentum tensor T_{ij} corresponding to the variation of Lagrangian L with respect to the Higg's field and the electromagnetic field given by

$$T_{ij} = \overline{T}_{ij} + E_{ij}, \quad (4)$$

where

$$\overline{T}_{ij} = \phi_{,i}\phi_{,j} - [\frac{1}{2}\phi_{,k}\phi^{,k} + V(\phi)]g_{ij} \quad (5)$$

with

$$\frac{1}{\sqrt{-g}}\partial_{,i}(\sqrt{-g}\partial^{,i}\phi) = \frac{-dV(\phi)}{d\phi} \quad (6)$$

and E_{ij} is the electromagnetic field given by Lichnerowicz [21] as

$$E_{ij} = \frac{1}{4\pi}[-F_{is}F_j^s + \frac{1}{4}g_{ij}F_{sp}F^{sp}], \quad (7)$$

where comma (,) and semicolon (;) indicate ordinary and covariant differentiation respectively. The function ϕ depends on t only due to homogeneity and F_{ij} is the Maxwell's electromagnetic tensor and units are taken such that

$$8\pi G = c = 1.$$

We assume that coordinates to be co-moving so that

$$v^1 = v^2 = v^3 = 0 \quad \text{and} \quad v^4 = 1. \quad (8)$$

Hence, we have, from equation (5)

$$\overline{T}_1^1 = \overline{T}_2^2 = \overline{T}_3^3 = \frac{-\dot{\phi}^2}{2} - V(\phi) \quad \text{and} \quad \overline{T}_4^4 = \frac{\dot{\phi}^2}{2} - V(\phi). \quad (9)$$

We assume that the magnetic field is due to an electric current produced along the x -axis. Thus, the magnetic field is in the yz -plane. Therefore, the F_{23} is the only non-vanishing component of F_{ij} and $h_1 \neq 0, h_2 = 0 = h_3 = h_4$. We also find that $F_{14} = 0 = F_{24} = 0 = F_{34}$ due to the assumption of infinite electrical conductivity.

The Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \quad (10)$$

and

$$F_{;j}^{ij} = 0, \quad i.e. \quad \frac{\partial}{\partial x^j} (F^{ij} \sqrt{-g}) = 0$$

together lead to the result

$$F_{23} = \text{const} = I \text{ (say).}$$

Hence the non-vanishing components of E_{ij} corresponding to the line element (1) are given by

$$\begin{aligned} E_1^1 &= \frac{I^2}{8\pi B^4 \sin^2 \theta}, \\ E_2^2 &= -\frac{I^2}{8\pi B^4 \sin^2 \theta}, \\ E_3^3 &= -\frac{I^2}{8\pi B^4 \sin^2 \theta}, \\ E_4^4 &= \frac{I^2}{8\pi B^4 \sin^2 \theta}. \end{aligned} \quad (11)$$

The Einstein's field equations (3) with the help of (9) and (11) for the metric (1) are given by

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} = -\frac{1}{2}\dot{\phi}^2 - V(\phi) + \frac{I^2}{8\pi B^4 \sin^2 \theta}, \quad (12)$$

$$\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -\frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{I^2}{8\pi B^4 \sin^2 \theta}, \quad (13)$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} = \frac{1}{2}\dot{\phi}^2 - V(\phi) + \frac{I^2}{8\pi B^4 \sin^2 \theta} \quad (14)$$

and the equation for the scalar field is

$$\ddot{\phi} + \dot{\phi} \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \right) = -\frac{dV(\phi)}{d\phi}. \quad (15)$$

Here the dot ($\dot{}$) overhead letter denotes differentiation with respect to t .

3 Inflationary Model

Stein-Schabes [22] has shown that Higg's field ϕ with potential $V(\phi)$ has a flat region and the field evolves slowly but the universe expands in an exponential

way due to the vacuum field energy. It is assumed that the scalar field will take sufficient time to cross the flat region so that the universe expands sufficiently to become homogeneous and isotropic on the scale of the order of the horizon size. Thus, we are interested here in inflationary solutions of the field equations; flat region is considered where the potential is constant, *i.e.*

$$V(\phi) = \text{const} = V_0 \quad (\text{say}). \quad (16)$$

The equation (15) on integration gives

$$\dot{\phi} = \frac{k_1}{AB^2}, \quad (17)$$

where k_1 is a constant of integration.

Since the field equations are highly non-linear, to get a determinate solution one can use a spatial law of variation of Hubble parameter proposed by Bermann [20], which yields a negative constant deceleration parameter model of the universe. Here we consider only negative constant deceleration parameter defined by

$$q = -\left(\frac{\ddot{a}a}{\dot{a}^2}\right) = \text{const}, \quad (18)$$

where $a = (AB^2)^{1/3}$ is the overall scale factor. Here the constant is taken as negative, since it is an accelerating model of the universe.

Equation (18) gives the solution

$$a(t) = (AB^2)^{1/3} = (ct + d)^{\frac{1}{1+q}}, \quad (19)$$

where c and d are constants of integration. This equation implies that the condition of expansion is $1 + q > 0$.

Differentiating Eq. (19), we get

$$\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} = \frac{3c}{(1+q)(ct+d)}. \quad (20)$$

Now from Eqs. (12), (13) and (15), we obtain

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} = -V_0 - \frac{I^2}{8\pi B^4 \sin^2 \theta}. \quad (21)$$

Eliminating \dot{B}/B from Eqs. (20) and (21) by using $\dot{A}/A = R$, we obtain

$$\dot{R} + \left[\frac{3c}{(1+q)(ct+d)}\right]R = -\left[V_0 + \frac{I^2}{M}\right], \quad (22)$$

where $M = 8\pi B^4 \sin^2 \theta$.

The solution of (22) yields

$$R = \frac{-(V_0 + I^2/M)}{c} \left(\frac{q+1}{q+4} \right) (ct+d) + l(ct+d)^{\frac{-3}{1+q}} \quad (23)$$

which on integration with $R = \frac{\dot{A}}{A}$ gives

$$A = m \exp \left\{ \frac{-(V_0 + I^2/M)}{2c^2} \left(\frac{q+1}{q+4} \right) (ct+d)^2 + \frac{l}{c} \left(\frac{q+1}{q-2} \right) (ct+d)^{\frac{q-2}{q+1}} \right\}. \quad (24)$$

Using (23) in Eq. (20), we obtain

$$B = n(ct+d)^{\frac{3}{2(q+1)}} \exp \left\{ \frac{(V_0 + I^2/M)}{4c^2} \left(\frac{q+1}{q+4} \right) (ct+d)^2 - \frac{l}{2c} \left(\frac{q+1}{q-2} \right) (ct+d)^{\frac{q-2}{q+1}} \right\}, \quad (25)$$

where l , m , and n are constants of integration.

Now using Eqs. (24) and (25) in Eq. (17) and integrating, we obtain

$$\phi = \frac{k_1}{c} \left(\frac{q+1}{q-2} \right) (ct+d)^{\frac{q-2}{q+1}} + k_2, \quad (26)$$

where k_2 is the constant of integration.

After a suitable choice of coordinates and constants of integration, the magnetized Kantowski-Sachs inflationary model corresponding to the solutions (24) and (25) takes the form

$$ds^2 = dt^2 - e^{-(V_0 + I^2/M)\delta t^2 + \frac{2}{\gamma} t^\gamma} dr^2 - t^{\frac{3}{1+q}} e^{\frac{1}{2}(V_0 + I^2/M)\delta t^2 - \frac{1}{\gamma} t^\gamma} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (27)$$

where $\delta = \frac{q+1}{q+4}$ and $\gamma = \frac{q-2}{q+1}$ ($m = n = c = l = 1$ and $d = 0$).

The scalar field ϕ gives

$$\phi = \frac{k_1}{\gamma} t^\gamma + k_2. \quad (28)$$

4 Some Physical and Kinematical Parameters of the Model

The model (27) represents an anisotropic magnetized Kantowski-Sachs inflationary cosmological model in general relativity when the scalar field is minimally coupled to the gravitational field in which the flat region of potential is

constant which is generally associated with vacuum energy. The model obtained in (27) is free from singularity.

The physical and kinematical parameters for the model (27) are the spatial volume V_3 , the expansion scalar θ , the shear scalar σ^2 and the Hubble parameter H , and have the following expressions:

$$\begin{aligned} V^3 &= t^{\frac{3}{1+q}}, \\ \theta &= \frac{3}{(1+q)t}, \\ \sigma^2 &= \frac{2}{3} \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = \frac{3}{2} \left[- \left(V_0 + \frac{I^2}{M} \right) \delta t + t^{\frac{-3}{q+1}} - \frac{1}{(q+1)t} \right]^2, \\ H &= \frac{\dot{a}}{a} = \frac{1}{(1+q)t}. \end{aligned}$$

We observe that for the model (27) the spatial volume increases with the cosmic time t when $1+q > 0$ and it becomes infinite for a large value of t . Thus inflationary scenario appears in general relativity for anisotropic Kantowski-Sachs space-time with massless scalar field. It is seen that the model starts with a big-bang at $t = 0$ in the presence of magnetic field and the expansion in the model decreases as time increases. The shear scalar σ^2 diverges at $t \rightarrow 0$. We can find that for large t the Hubble parameter vanishes and it becomes infinite at $t \rightarrow 0$. The Higg's field ϕ becomes constant for $t = 0$ and for large t it diverges. Since $\lim_{t \rightarrow \infty} \frac{\sigma^2}{\theta} \neq 0$, the model does not isotropize for large values of t .

5 Conclusions

In this paper we present the magnetized Kantowski-Sachs inflationary cosmological model in the presence of massless scalar field with a flat region of constant potential in general relativity. It is observed that the model obtained is free from singularity and does not approach isotropy for large t . The model in the absence of magnetic field also starts with a big-bang at $t = 0$ and the expansion in the model decreases as time increases. Here we can conclude that the magnetic field is not so much effective to increase the inflation rate in its early stages but later on it supports the inflation through giving an increase in the rate of inflation. In view of the fact that there is an increasing interest, in the recent years, in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology, the inflationary model obtained here has a considerable astrophysical significance.

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