Bulg. J. Phys. **37** (2010) 65–89

Exact Solutions of Teukolsky Master Equation with Continuous Spectrum

R.S. Borissov, P.P. Fiziev

Physics Department, University of Sofia, 5 James Bourchier Blvd., 1164 Sofia, Bulgaria

Received *3 August 2010*

Abstract. Weak gravitational, electromagnetic, neutrino and scalar fields, considered as perturbations on Kerr background satisfy Teukolsky Master Equation. The two non-trivial equations obtained after separating the variables are the polar angle equation and the radial equation. We solve them by transforming each one into the form of a confluent Heun equation. The transformation depends on a set of parameters, which can be chosen in a such a way, so the resulting angular and radial equations separately have simple polynomial solutions for neutrino, electromagnetic, and gravitational perturbations, provided some additional conditions are satisfied. Remarkably there exists a class of solutions for which these additional conditions are the same for both the angular and the radial equations for spins $|s| = 1/2$ and $|s| = 1$. As a result the additional conditions fix the dependence of the separation constant on the angular frequency but the frequency itself remains unconstrained and belongs to a continuous spectrum.

PACS number: 04.20.Ex

1 Introduction

Fields of various types – scalar, neutrino, electromagnetic, and gravitational – have been extensively studied as perturbations to known solutions of Einstein's equations for configurations with spherical and cylindrical symmetry. The fields considered are weak in the sense that we can neglect the influence of their stress-energy tensor on the background metric. Regge and Wheeler [1] and Zerilli [2,3] were the first to study the linear response of Schwarzschild solution of Einstein's equations to perturbations. In order to study Kerr metric perturbations Teukolsky [4–7] analyzed the components of Weyl tensor, using Newman-Penrose formalism [15]. (For a detailed extended presentation see [12].). As a result one obtains the Teukolsky Master Equation, which describes the dynamics of various fields of different spins as perturbations to Kerr metric. In recent years there is an increased interest on the subject [13–16], mostly aimed at studying the quasi-normal modes. Another problem analyzed via Teukolsky's equations is related to the decaying of the various fields present during a gravitational collapse at very late times at large distances – the so called late-time tails. All these investigations however, are performed via indirect, approximate methods [13, 14, 16–18].

On the other hand, already for quite some time it has been recognized in the literature [19–22] that Regge-Wheeler and Teukolsky's equations can be transformed into the form of a confluent Heun equation [23–28]. The reason there has not been much attention paid to the Heun-type solutions is that they are not completely analyzed and, in general, difficult to work with. Some basic classes of exact solutions to Rege-Wheeler equation in terms of special solutions to the confluent Heun equation – the so-called confluent Heun functions (see the Appendix), were described recently and were used for finding solutions to a number of physical problems [30–32].

Following the articles [30–32] we continue with the application of the confluent Heun functions to Teukolsky's equations. The first results, presented in [33–38], were very encouraging and drew special attention to the solutions in terms of the confluent Heun polynomials [23–29] (see the Appendix). It should be emphasized that long time ago in [19, 20] it was recognized by Baldin, Pons, and Marcilhacy that the conditions for polynomial solutions to Heun equations lead to polynomial solutions to Teukolsky's equations in a generalized sense, i.e. polynomials multiplied by non-polynomial factors which are elementary functions. Having in mind the general description of all 256 classes of factorized solutions to Teukolsky Master Equation [39, 40] we intend to focus on the mathematical properties of some of them and study various physical applications.

The general description of all polynomial solutions of Teukolsky Master Equation was given for the first time in [39, 40]. These fall into two different classes. For the first class, the first polynomial condition (A.9), called the δ_N-condition in [29,39,40], is automatically satisfied. For waves of spin $|s|$ this condition fixes only the degree $(N+1) = 2|s|$ of the second polynomial condition $\Delta_{N+1} = 0$. For the second class of polynomial solutions the δ_N-condition is fulfilled only for certain complex frequencies ω_N which belong to definite equidistant discrete spectra. For the two classes the second polynomial condition $\Delta_{N+1} = 0$ defines an algebraic equation of degree $2|s|$ for the second separation constant: $E_m = E_m(\omega)$, and $E_m = E_m(\omega_N)$, correspondingly.

Here we are considering only polynomial solutions of the first class. Thus an independent derivation of the specific relations, valid only for the first class of polynomial solutions becomes possible. It is based on a direct check of the two necessary conditions (A.9) and (A.10), which together are sufficient to ensure the polynomial character of the solutions (see the Appendix).

Below we present an independent derivation of the first class polynomial solution both for Teukolsky's angular and radial equations using the notations of reference [28]. This notation has some advantages since it simplifies significantly the form of the δ_N-condition. The correspondence between the notation of [28] and the notation used in [25, 26, 29–40] and in the computer application Maple is described in Section 9.4 of the Appendix.

In the present paper we consider a specific type of evolution of week fields with spin $|s| = 1/2$, 1 and 2 on Kerr background. The solutions studied here are double polynomial solutions that describe one-way waves of corresponding spins, the so-called total transmission modes. These are factorized solutions to Teukolsky Master Equation, in which the solutions both of the angular and the radial equations (of the same spin weight s) belong to the corresponding first classes of polynomial solutions, introduced in [39, 40]. Here we show that these solutions yield a complex one-parameter continuous spectrum of the frequency ω and derive the explicit form of the separation constant E in the various cases. Finally we discus some overall solutions of Teukolsky Master Equation, constructed making use only of these continuous spectrum solutions.

To the best of our knowledge this is the first time when for Teukolsky Master Equation exact solutions with continuous spectrum are presented for a specific boundary problem. An interesting observation is that continuous spectrum emerges only for neutrino and electromagnetic waves, because of the simultaneous fulfillment of the polynomial condition both for the angular and the radial Teukolsky equations. We have to stress that such a simultaneous fulfillment is not in place for gravitational waves. The physical consequences of this mathematical result may be deep and very important. Its roots can be traced back to some results, originally obtained in [7] and developed further in [12]. We present here the mathematical basis, needed for further developments in this direction.

In the next section we start by reminding the procedure for separation of the variables in Teukolsky Master Equation via factorization of the solutions and the corresponding basic results. In section 3 we present the general scheme for transforming Teukolsky's radial equation (TRE) into the one of the many known "canonical" forms of the confluent Heun equation [23–29][1], namely into the so-called non-symmetrical canonical form. We show that for specific values of the indices of the regular singular points [41] the first condition for polynomial solution to TRE in the form of a confluent Heun equation is automatically attained. We impose the second condition for having a polynomial solution and obtain the value of the separation parameter E as function of the frequency ω. In section 4 we continue by presenting the transformation to non-symmetrical canonical Heun form of Teukolky's angular equation (TAE) and again show that for corresponding specific choice of the indices of its regular singular points we

[1] In the literature there still does not exist a commonly accepted standard form both of the five classes of Heun equations and of the corresponding Heun functions. Different possible forms are in use, since in different applications the authors prefer the form, which is most suitable for their specific needs.

obtain again a polynomial solution. In order to achieve this result we derive the explicit form of the second polynomial condition and arrive at the result that in some cases it is the same as for the radial equation. Thus we find a simultaneous fulfillment of the polynomial conditions for the angular and for the radial Teukolsky's equations for perturbations with spin $|s| = 1/2$ and 1. In section 5 we discuss why such a simultaneous fulfillment of the polynomial conditions is not possible for gravitational waves ($|s| = 2$). In section 6 we present the basic properties of the overall solutions to the Teukolsky Master Equation constructed only from the factorized solutions with continuous spectrum. We show that these solutions describe one-way collimated waves, which may be regular along the rotational axes, despite the singular character of the polynomial solutions of the angular Teukolsky equation. In mathematical sense these solutions form a natural orthogonal basis of singular functions for integral representation of physically meaningful solutions. In the conclusion we give a brief summary and ideas for future studies on the matter.

In an Appendix some basic properties of the confluent Heun equation and its solutions and different forms are presented for the reader's convenience.

2 Spin Weight s Fields on Kerr Background

In this section we present some basic results of Teukolsky's approach [4–7] to the perturbations of spin $|s|$ of Kerr vacuum solution for the metric of a rotating black hole. In Boyer-Lindquist coordinates the metric is given by [12, 42]:

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left[r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 \,. \quad (1)$$

Here M is the Keplerian mass of the rotating black hole and a is its angular momentum per unit mass. Also, Δ and Σ are defined in the usual way:

$$\Delta \equiv r^2 - 2Mr + a^2, \qquad \Sigma \equiv r^2 + a^2\cos^2\theta \,. \quad (2)$$

The dynamics of a massless field $\Psi = \Psi(t, r, \theta, \phi)$ with spin weight s is described by Teukolsky Master Equation:

$$\Big[\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\Big]\frac{\partial^2\Psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\Psi}{\partial t\partial\phi} + \Big[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\Big]\frac{\partial^2\Psi}{\partial\phi^2} - \Delta^{-s}\frac{\partial}{\partial r}\Big(\Delta^{s+1}\frac{\partial\Psi}{\partial r}\Big) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Psi}{\partial\theta}\Big) - 2s\Big[\frac{a\,(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\Big]\frac{\partial\Psi}{\partial\phi} - 2s\Big[\frac{M\left(r^2-a^2\right)}{\Delta} - r - ia\cos\theta\Big]\frac{\partial\Psi}{\partial t} + (s^2\cot^2\theta - s)\Psi = 0 \,. \quad (3)$$

In the above equation we have the following expressions for Ψ:

- For $s = 1/2$, $\Psi = \chi_0$ and for $s = -1/2$, $\Psi = \rho^{-1}\chi_1$, where χ_0 and χ_1 represent the two components of the neutrino spinor in Newman-Penrose formalism.
- For $s = 1$, $\Psi = \varphi_0$ and for $s = -1$, $\Psi = \rho^{-2}\varphi_2$, where φ_0 and φ_2 are Maxwell tensor tetrad components in Newman-Penrose formalism.
- In the gravitational case for $s = 2$, $\Psi = \psi_0$ and for $s = -2$, $\Psi = \rho^{-4}\psi_4$, where ψ_0 and ψ_4 are Weyl tensor Newman-Penrose tetrad components.

In all cases $\rho = -1/(r - ia\cos\theta)$ and the choice of the Kinnersley tetrad is assumed. In order to separate the variables following [4–7] we set

$$\Psi(t, r, \theta, \phi) = \frac{1}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\sum_{m=-\infty}^{\infty}\Psi_m(\omega; t, r, \theta, \phi)d\omega\,, \tag{4}$$

where $\Psi_m(\omega; t, r, \theta, \phi) = e^{im\phi}e^{-i\omega t}S_m(\omega;\theta)R_m(\omega;r)$, and ϵ is a parameter defining the contour of integration. The azimuthal number has values $m = 0, \pm1, \pm2, \dots$ for integer spin, or $m = \pm1/2, \pm3/2, \dots$ for half-integer spin [8–10, 40].

Thus we are looking for an integral representation with a *factorized* kernel $\Psi_m(\omega; t, r, \theta, \phi)$. For the unknown factors $S_m(\omega;\theta)$ and $R_m(\omega;r)$ we obtain respectively Teukolsky's angular equation (TAE):

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)S_m(\omega;\theta) + \Big(a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta - \frac{m^2 + s^2 + 2ms\cos\theta}{\sin^2\theta} + E_m\Big)S_m(\omega;\theta) = 0\,, \tag{5}$$

and Teukolsky's radial equation (TRE)

$$\Delta^{-s}\frac{d}{dr}\Big(\Delta^{s+1}\frac{d}{dr}\Big)R_m(\omega;r) + \Big(\frac{K^2 - 2is\,(r-M)\,K}{\Delta} + 4is\omega r - \lambda_m\Big)R_m(\omega;r) = 0\,, \tag{6}$$

where $K \equiv (r^2 + a^2)\omega - am$ and $\lambda_m \equiv E_m + a^2\omega^2 - 2am\omega - s(s+1)$ is the separation "constant"[2]. These are the two equations, some special solutions of which we will study in detail in our paper.

[2] In the literature often another form of the separation constant is used, namely $A_m = E_m - s(s+1)$ so λ_m can also be written as $\lambda_m \equiv A_m + a^2\omega^2 - 2am\omega$.

3 Solutions to TRE

3.1 Transforming TRE into the Non-Symmetrical Canonical Form of Heun Equation

Both equations (5) and (6) are second order ordinary differential equations with two regular singular points at finite value of the independent variable and one irregular singularity at infinity. The most general equation with such properties is the confluent Heun equation [28]. Thus both TRE and TAE are specific cases of confluent Heun equations. At this point we continue by transforming the radial equation (6) into one of the forms of the confluent Heun equation, the non-symmetrical canonical form (A.1). In order to do so we apply the so-called *s*-homotopic transformation to (6) by setting[3]

$$R_m(\omega; r) = (r - r_+)^{\xi}(r - r_-)^{\eta} e^{\zeta r} H(r)\,, \tag{7}$$

where r_+ and r_- are the event and Cauchy horizons of a Kerr black hole defined by $r_\pm = M \pm \sqrt{M^2 - a^2}$ and ξ, η (the indices of the regular singularities) and ζ are parameters to be determined. By substituting (7) into (6) and after some straightforward algebra we arrive at the following equation for $H(r)$:

$$\begin{aligned}
&\frac{d^2 H(r)}{dr^2} + \left(\frac{2\xi + s + 1}{r - r_+} + \frac{2\eta + s + 1}{r - r_-} + 2\zeta\right)\frac{dH(r)}{dr}\\
&\quad + \frac{1}{(r - r_+)(r - r_-)}\{[4\omega^2 M + 2\zeta(\xi + \eta + s + 1) + 2is\omega]r\\
&\quad + (\xi + \eta)^2 + (\xi + \eta) + 2s(\xi + \eta) - 2\zeta(\xi r_- + \eta r_+) - 2s\zeta M - 2\zeta M\\
&\quad - 2a\omega m + 4\omega^2 M^2 - 2is\omega M - \lambda_m\}H(r) = 0\,.
\end{aligned}$$

In order to obtain this form of the equation we had to fix the parameters ξ, η and ζ. For them we end up with quadratic equations and thus with pairs of possible expressions, namely:

$$\xi_1 = ia_+(\omega - m\Omega_+), \qquad \xi_2 = -s - ia_+(\omega - m\Omega_+)\,, \tag{8}$$

$$\eta_1 = -s + ia_-(\omega - m\Omega_-), \qquad \eta_2 = -ia_-(\omega - m\Omega_-)\,, \tag{9}$$

and

$$\zeta_1 = i\omega, \qquad \zeta_2 = -i\omega\,, \tag{10}$$

where we have set

$$a_\pm = \frac{2Mr_\pm}{r_+ - r_-}, \qquad \Omega_\pm = \frac{a}{2Mr_\pm}\,. \tag{11}$$

[3] This anzatz was used for the first time for analytical and numerical studies of the problem at hand by Leaver [18], [43, 44].

Any one of the eight possible triplets of expressions from (8), (9), and (10) leads to a Heun equation in the desired form with different parameters. In order to determine these parameters it is necessary to perform one more step and to introduce dimensionless variables instead of r. In order to keep the symmetries in the problem manifest it is best to set different variables for positive and for negative spin weights, namely for $s = 1/2, 1, 2$ and for $s = -1/2, -1, -2$ we will have respectively:

$$ {}_{+}z = \frac{r_+ - r}{r_+ - r_-}, \qquad {}_{-}z = \frac{r - r_-}{r_+ - r_-}. $$

Note that from now on in the paper we will denote the sign of the spin weight (and when there might be a confusion the spin weight itself) by a subscript to the left of the variable. Also always the signs "+", "-", or $\pm$ used as subscripts to the right from the variables will be related to the two singularities in the radial equation, namely to the event and the Cauchy horizons. After performing these adjustments we end up with a Heun equation in the non-symmetrical standard form (A.1):

$$ \frac{d^2H}{d({}_{\pm}z)^2} + \left(4({}_{\pm}p) + \frac{{}_{\pm}\gamma}{{}_{\pm}z} + \frac{{}_{\pm}\delta}{{}_{\pm}z - 1}\right)\frac{dH}{d({}_{\pm}z)} + \frac{4({}_{\pm}\alpha)({}_{\pm}p)({}_{\pm}z) - ({}_{\pm}\sigma)}{({}_{\pm}z)(({}_{\pm}z) - 1)}H = 0 \quad (12) $$

with the parameters given by the following expressions:

$$ \begin{gathered} {}_{\pm}p = \mp(r_+ - r_-)\zeta/2, \\ {}_{+}\gamma = {}_{-}\delta = 2\xi + s + 1, \qquad {}_{-}\gamma = {}_{+}\delta = 2\eta + s + 1, \\ {}_{\pm}\alpha = 2M\omega^2\zeta^{-1} + \xi + \eta + s + 1 + is\omega\zeta^{-1}\,, \end{gathered} \quad (13) $$

and

$$ \begin{aligned} {}_{\pm}\sigma = -2\zeta r_{\pm}\Big(\frac{2M\omega^2}{\zeta} + \xi + \eta + s + 1 + \frac{is\omega}{\zeta}\Big) \\ - \Big[(\xi+\eta)^2 + (\xi+\eta) + 2s(\xi+\eta) - 2\zeta(\xi r_- + \eta r_+) \\ - 2s\zeta M - 2\zeta M - 2a\omega m + 4\omega^2M^2 - 2is\omega M - \lambda_m\Big]\,, \end{aligned} $$

for any choice of a triplet ξ, η, ζ. In general there exist pairs of solutions to Heun equation expressed as appropriate power series about each one of the singularities. The solutions of the equation (12) can be expressed with the use of the so-called Frobenius solutions $Hc^{(r)}({}_{\pm}p, {}_{\pm}\alpha, {}_{\pm}\gamma, {}_{\pm}\delta, {}_{\pm}\sigma; {}_{\pm}z)$. There exists a multitude of other solutions to Heun equation which can be obtained from these Frobenius solution by interchanging the finite singular points or performing appropriate *s*-homotopic transformations (see the Apendix).

3.2 Polynomial Solutions

Thus we have completed the transformation of TRE to the non-symmetrical form of the confluent Heun equation and we can formally identify the solutions which are given by specific Heun functions. There exists though a special case, in which the confluent Heun equation admits polynomial solutions (See the Appendix). This special case depends on the values of ${}_{\pm}\alpha$ in (13) and another, more involved condition on the parameters in the Heun equation. If the parameters ${}_{\pm}\alpha$ are equal to negative integer numbers or zero and if we can solve the second condition then we will have polynomial solutions to Heun equation. At this point we can make the following observation regarding the possible values of ${}_{\pm}\alpha$ in (13): In the cases of positive spin weights if we pick the values ξ_2, η_1, and ζ_2 from (8), (9), and (10), then we obtain ${}_{+}\alpha = 1 - 2s$, which equals 0 for neutrino, -1 in the electromagnetic case and -3 in the gravitational case. Similarly, for $s = -1/2$, $s = -1$, or $s = -2$ we have to choose ξ_1, η_2, and ζ_1 and we will get ${}_{-}\alpha = 1 + 2s$, which again gives $\alpha = 0, -1$ or -3 for the neutrino, for the electromagnetic, and for gravitational case respectively. Thus we have identified particular combinations of the values of the parameters ξ, η, and ζ for which we may expect to find polynomial solutions of the radial equation for $|s| = 1/2, 1, 2$ since the necessary condition $\alpha = -N$, N-integer, is satisfied. Also, it can be seen easily that for $s = 0$ we obtain ${}_{\pm}\alpha = 1$ so there is no polynomial solution for scalar fields. The second condition will depend on the value of N. It is an algebraic equation of order $N + 1$ and leads to polynomial solutions to Heun equations of order N. Thus we expect that in the neutrino case the polynomial solutions, if they exist, are simply constants, in the electromagnetic case - linear functions, and in the gravitational case the solutions are cubic polynomials. We will consider here in more detail only the electromagnetic case. The results for $s = \pm 1/2$ can be easily obtained using the same procedure. The polynomial solutions to TRE with spin $|s| = 2$ were described long time ago in quite a different setting by Chandrasekhar [45]. For a more recent treatment see [46].

3.3 Polynomial Solution to TRE for Spin $|s| = 1$

In order to find a polynomial solution to TRE we have to impose in addition to the fact that ${}_{\pm}\alpha = -N$ the condition (A.10). In this case we are looking for an expansion about the singular point at infinity. The second polynomial condition for electromagnetic perturbations has the form $g_0^{(r)} g_1^{(r)} = h_1^{(r)} f_0^{(r)}$ with the coefficients from the three-term relation (A.8) given by $g_0^{(r)} = -\sigma - 4p + \gamma + \delta, g_1^{(r)} = -\sigma, f_0^{(r)} = -4p, h_1^{(r)} = -\gamma$. Provided we make the above mentioned choices rendering ${}_{\pm}\alpha = -1$ we obtain the following values for the remaining parameters in the Heun equation: ${}_{\pm}p = (i/2)\omega(r_+ - r_-), {}_{\pm}\gamma = \pm 2ia_-(\omega - m\Omega_-), {}_{\pm}\delta = \mp 2ia_+(\omega - m\Omega_+), {}_{\pm}\sigma = {}_{\pm}E_m + a^2\omega^2 - 2a\omega m \mp$

$2i\omega r_\pm$. This second polynomiality condition essentially fixes ${}_\pm\sigma$. It leads to a quadratic equation for ${}_\pm\sigma$ with solutions ${}_\pm\sigma_1 = \mp 2i\omega r_\pm + 2\sqrt{a\omega(a\omega - m)}$ and ${}_\pm\sigma_2 = \mp 2i\omega r_\pm - 2\sqrt{a\omega(a\omega - m)}$ (recall that the $\pm$ sign to the left of σ refers to the sign of the spin weight, while the subscripts 1 and 2 number the solutions to the quadratic equation).

The first important result from solving the second condition for having polynomial solutions is that we find the dependence of the separation "constant" E_m on the frequency ω, which is the same for both $s = +1$ and for $s = -1$ [39, 40] and is given by

$$
\begin{aligned}
{}_{\pm 1}E_m(a\omega)_1 &= -a^2\omega^2 + 2a\omega m + 2\sqrt{a\omega(a\omega - m)} \\
{}_{\pm 1}E_m(a\omega)_2 &= -a^2\omega^2 + 2a\omega m - 2\sqrt{a\omega(a\omega - m)}\,.
\end{aligned}
$$

By returning to the r variable we arrive at the following expressions for the polynomial solutions of the Heun equation for $s = 1$:

$$
({}_{+1}H_m(\omega; r))_{1,2} = \frac{1}{r_+ - r_-}\left(-r \pm \frac{i}{\omega}\sqrt{a\omega(a\omega - m)}\right)\,. \qquad (14)
$$

Similarly, for $s = -1$ we have:

$$
({}_{-1}H_m(\omega; r))_{1,2} = \frac{1}{r_+ - r_-}\left(r \pm \frac{i}{\omega}\sqrt{a\omega(a\omega - m)}\right)\,. \qquad (15)
$$

Putting together (7) with the specific values of ξ, η, and ζ and the polynomial solutions of Heun equations, the solutions to the TRE for $s = 1$ and for $s = -1$ can be written (modulo normalizing constants) respectively as:

$$
({}_{+1}R_m(\omega; r))_{1,2} = \frac{e^{-i\omega r_*}}{\Delta}\left(\frac{r - r_+}{r - r_-}\right)^{\frac{ima}{r_+ - r_-}} ({}_{+1}H_m(\omega; r))_{1,2}\,, \qquad (16)
$$

and

$$
({}_{-1}R_m(\omega; r))_{1,2} = e^{i\omega r_*}\left(\frac{r - r_+}{r - r_-}\right)^{-\frac{ima}{r_+ - r_-}} ({}_{-1}H_m(\omega; r))_{1,2}\,, \qquad (17)
$$

where

$$
r_* = r + a_+ \ln|r - r_+| - a_- \ln|r - r_-|
$$

is the "tortoise" coordinate and $a_\pm$ are defined in (11). Note that these exact solutions for $|s| = 1$ are presented here for the first time in explicit form. A similar form of polynomial solutions to TRE in the case $|s| = 2$ can be found in [46].

Using the orthogonality relations of Heun polynomials [28] it can be shown that in terms of the intermediate variables ${}_sz = {}_\pm z$ we have:

$$\int_{-\infty}^{0} \Delta^s ({}_sR_m(\omega;{}_sz))_j ({}_sR_m(\omega;{}_sz))_l d({}_sz) = 0, \qquad j \neq l \qquad j,l = 1,2\,, \tag{18}$$

where Δ is the standard factor from the Kerr metric, defined in (2). The behavior of these solutions at infinity and at the event horizon can be readily determined form (16) and (17). First, both solutions in (16) with $s = 1$ have the behavior:

$$({}_{+1}R_m(\omega;r))_{1,2} \sim \begin{cases} r^{-1}e^{-i\omega r_*} & r \to \infty \quad (r_* \to \infty) \\ \Delta^{-s}e^{-i\varpi r_*} & r \to r_+ \ (r_* \to -\infty) \end{cases}, \tag{19}$$

while both solutions in (17) with $s = -1$ behave like

$$({}_{-1}R_m(\omega;r))_{1,2} \sim \begin{cases} r^{-(2s+1)}e^{i\omega r_*} & r \to \infty \quad (r_* \to \infty) \\ e^{i\varpi r_*} & r \to r_+ \ (r_* \to -\infty) \end{cases}, \tag{20}$$

where in the expressions for the behavior at r_+ we have set $\varpi = \omega - m\Omega_+$. Thus we found exact solutions to TRE, the nature of which depends on the relative sign between ω and ϖ, in agreement with the general analysis in [4]- [6]. When ω and ϖ have the same sign then the solutions we found describe one-way waves travelling from r_+ to infinity or in the opposite direction. The solutions with ω and ϖ with opposite signs describe either waves travelling towards r_+ and towards infinity or leaving from r_+ and coming from infinity.

3.4 Polynomial Solution for Spin $|s| = 1/2$

This case follows along the same lines like the electromagnetic perturbations but is simpler since $\alpha = 0$. This leads to $c_1^{(r)} = 0$ in (A.8) which corresponds to a constant solution to the Heun equation. This implies that $\sigma = 0$ and thus determines the dependence of the separation constant E from ω. The result is [39, 40]

$${}_{\pm\frac{1}{2}}E_m(a\omega) = -a^2\omega^2 + 2a\omega m - \frac{1}{4}.$$

The solutions to TRE for $s = \pm\frac{1}{2}$ are presented here for the first time:

$${}_{\frac{1}{2}}R_m(\omega;r) = \frac{e^{-i\omega r_*}}{\sqrt{\Delta}} \left(\frac{r - r_+}{r - r_-}\right)^{\frac{ima}{r_+ - r_-}} {}_{\frac{1}{2}}H_m(\omega) \tag{21}$$

$${}_{-\frac{1}{2}}R_m(\omega;r) = e^{i\omega r_*} \left(\frac{r - r_+}{r - r_-}\right)^{-\frac{ima}{r_+ - r_-}} {}_{-\frac{1}{2}}H_m(\omega)\,, \tag{22}$$

where ${}_{\frac{1}{2}}H_m(\omega)$ and ${}_{-\frac{1}{2}}H_m(\omega)$ are constants in r. Note that these solutions satisfy the same orthogonality relations as those in (18) and have the same behavior at infinity and at r_+ as those given in (19) and (20).

4 Solutions to TAE

4.1 Transformation of TAE into the Form of a Heun Equation

In order to transform TAE (5) into the form of a Heun equation we follow the same procedure as with the radial one. Following [47–51] we set

$$S_m(\omega;u) = (1-u)^{\mu_1}(1+u)^{\mu_2} e^{\nu u} T_m(\omega;u)\,, \tag{23}$$

where $u = \cos\theta$ and μ_1 and μ_2 are the indices of the regular singular points at $\theta = 0$ and $\theta = \pi$. We plug (23) into (5) and obtain an equation for $T_m(\omega;u)$. Imposing the condition that the equation for $T_m(\omega;u)$ has the form (A.1) leads to a system of quadratic algebraic equations for the parameters μ_1, μ_2, and ν. Their solutions are given by

$$\mu_1 = \pm\frac{m+s}{2}, \qquad \mu_2 = \pm\frac{m-s}{2}, \qquad \nu = \pm a\omega\,. \tag{24}$$

For these values of μ_1, μ_2, and ν the equation for $T_m(\omega;u)$ is given by

$$\begin{aligned}\frac{d^2T_m(\omega;u)}{du^2} &+ \left(\frac{2\mu_1+1}{u-1} + \frac{2\mu_2+1}{u+1} + 2\nu\right)\frac{dT_m(\omega;u)}{du}\\ &+ \frac{1}{(u-1)(u+1)}\left\{2\nu\left(\mu_1+\mu_2+1+\frac{a\omega s}{\nu}\right)u - \Big[E_m + a^2\omega^2\right.\\ &\left.\quad - (\mu_1+\mu_2)^2 - (\mu_1+\mu_2) - 2\nu(\mu_1-\mu_2)\Big]\right\}T_m(\omega;u) = 0\,. \end{aligned} \tag{25}$$

In order to complete the transformation we introduce new independent variables (the reason for the specific form of this transformation will become clear latter) $u \mapsto {}_+x$ for $s = +1/2, 1, 2$ and $u \mapsto {}_-x$ for $s = -1/2, -1, -2$, where ${}_\pm x = (1 \pm u)/2$. It is important to notice that these two independent variables are related by the transformation $\theta \mapsto \pi - \theta$. Thus we arrive at the following specific non-symmetric canonical form of a confluent Heun equation:

$$\begin{aligned}\frac{d^2T_m(\omega;{}_\pm x)}{d({}_\pm x)^2} &+ \left(\frac{{}_\pm\gamma}{{}_\pm x} + \frac{{}_\pm\delta}{{}_\pm x - 1} + 4({}_\pm p)\right)\frac{dT_m(\omega;{}_\pm x)}{d({}_\pm x)}\\ &+ \frac{4({}_\pm p)({}_\pm\alpha)({}_\pm x) - ({}_\pm\sigma)}{{}_\pm x({}_\pm x - 1)}T_m(\omega;{}_\pm x) = 0\,, \end{aligned} \tag{26}$$

with the identification ${}_\pm p = \pm\nu$, ${}_+\gamma = {}_-\delta = 2\mu_2 + 1$, ${}_-\gamma = {}_+\delta = 2\mu_1 + 1$, and for both signs we have

$${}_\pm\alpha = \mu_1 + \mu_2 + 1 + \frac{a\omega s}{\nu}$$

$$
{}_{\pm}\sigma = -2\nu({}_{\pm}\alpha) + E_m + a^2\omega^2 \\
- (\mu_1+\mu_2)^2 - (\mu_1+\mu_2) - 2\nu(\mu_1-\mu_2)\,. \quad (27)
$$

The solutions of equation (26) can be expressed with the use of the Frobenius solutions $Hc^{(a)}(p,\alpha,\gamma,\delta,\sigma;{}_{\pm}x)$ and all other solutions obtained from it by interchanging the finite singular points or performing appropriate *s*-homotopic transformations (see the Appendix). Thus we obtained the needed solutions to the Heun equation obtained from TAE[4]. In the general case these are given by two sets of infinite series about each one of the regular singularities. For more detailed description of all local solutions to the angular Teukolsky equation see [39, 40]. With these we can build the solutions to TAE.

For each choice of a triple μ_1, μ_2, and ν we get a different solution to TAE. There are two special cases though: First, we can choose μ_1, μ_2, and ν in such a way so to obtain solutions that are regular at both $\theta=0$ and $\theta=\pi$. This choice leads to a well studied Sturm-Liouville eigenvalue problem [**?**] for the spin-weighted spheroidal wave functions $S_{lm}(\omega;\theta)$ [52] and the separation constant $E_{lm} = E_{lm}(a\omega)$, which for fixed s, m, and $a\omega$ are labelled by an additional integer l. The eigenfunctions $S_{lm}(\omega;\theta)$ are complete and orthogonal on $0\le\theta\le\pi$ for each set s, m, and $a\omega$. In the case $s=0$, $S_{lm}(\omega;\theta)$ are the spheroidal wave functions. When $a\omega=0$, the eigenfunctions are the spin-weighted spherical harmonics ${}_sY_l^m = {}_sS_l^m(\theta)e^{im\phi}$. In the general case, as it is shown for example in [49] the function $T_m(\omega;z)$ can be expanded as a series of Jacobi polynomials in the case of an integer spin, and to their spin-weighted generalizations for half-integer spins [8–10], and one can obtain the separation constant λ_{lm} as a power series in $a\omega$.

The second special case corresponds to polynomial solutions to TAE [39, 40]. Let us choose $\mu_1 = -(m+s)/2, \mu_2 = (m-s)/2, \nu = -a\omega$. This choice leads in (27) to $\alpha = 1-2s$, which for $s=1/2$, $s=1$ and $s=2$ is zero or a negative integer number. This means that the condition (A.9), which is necessary for having a polynomial solution (a constant in the case $\alpha=0$) of the confluent Heun equation is met for these values of the spin weight s. The case of zero spin is again excluded by this condition. If instead we have $s=-1/2$, $s=-1$ or $s=-2$, we must choose $\mu_1=(m+s)/2, \mu_2=-(m-s)/2, \nu=a\omega$, and this will give us $\alpha=2s+1$, which in this case again is zero or a negative integer number. We have to emphasize that this result comes at a price - depending on the specific values of m and s either one or the other of the pre-factors in (23) diverge at the corresponding singular point. This means that for the cases of neutrino, electromagnetic and gravitational perturbations we eventually may write down the solutions to TAE into the form of diverging at some of the singularities pre-factor multiplied by a polynomial expression, simultaneously regular at both

[4] The Tomé asymptotic solution at infinity $Hc^{(r)}(p,\alpha,\gamma,\delta,\sigma;x)$ is not of physical significance in the case of angular equation (25), because here we are interested only in solutions on the interval $u\in(-1,1)$.

regular singular points. Again, the neutrino case is relatively simple and can be easily deduced from the electromagnetic one, which we will present in detail.

4.2 Polynomial Solutions for Perturbations with Spin $|s| = 1$

Since (A.9) is satisfied we can obtain a polynomial solution of (26) by imposing as an additional (already sufficiency) condition (A.10). In our case (A.10) translates into an ω dependence of the separation constant E_m, which enters in the parameter σ from (27).

In this case $\alpha = -1$, so we will have $c_2^{(a)} = 0$ in (A.10). Because of the specific choice we made when introducing ${}_{\pm}x$ in this case we have for both positive and negative spin weights the same values of the parameters in the Heun equation: $p = -a\omega, \alpha = -1, \gamma = m, \delta = -m, \sigma = E_m + a^2\omega^2 + 2a\omega - 2a\omega m$. This means that we obtain the same Heun equation for both $s = 1$ and $s = -1$ with Frobenius solutions $Hc^{(a)}(p, \alpha, \gamma, \delta, \sigma; {}_{\pm}x)$. The only difference between the two cases is in the definition of the independent variable ${}_{\pm}x = (1 \pm \cos\theta)/2$. Thus in this section we will look for a solution only for the case $s = 1$ and will obtain the solution for $s = -1$ by performing in the final results the transformation $\theta \mapsto \pi - \theta$. In terms of the coefficients from the three term relation (A.6) the sufficiency condition for $m \neq 0$ has the form $g_0^{(a)} g_1^{(a)} = h_1^{(a)} f_0^{(a)}$, where $g_0^{(a)} = -\sigma, g_1^{(a)} = 4a\omega - \sigma, f_0^{(a)} = -m, h_1^{(a)} = 4a\omega$. Since this condition leads to a quadratic equation for σ we obtain pairs of solutions. For both $s = +1$ and $s = -1$ we get the same expressions $\sigma_1 = 2a\omega + 2\sqrt{a\omega(a\omega - m)}$ and $\sigma_2 = 2a\omega - 2\sqrt{a\omega(a\omega - m)}$. Returning back to the original variables of the angular equation we can write down the following expressions [37, 38], for the solutions for $s = +1$ (for $m \neq 0$):

$$({}_{+1}S_m(\omega;\theta))_{1,2} = \frac{e^{-a\omega\cos\theta}}{\sin\theta}\left(\cot\frac{\theta}{2}\right)^m ({}_{+1}T_m(\omega;\theta))_{1,2}\,, \tag{28}$$

and we have restored the pre-subscripts denoting the two different spin weights. The polynomial parts of the solutions for $m \neq 0$ are:

$$({}_{+1}T_m(\omega;\theta))_1 = 1 - \frac{2a\omega + 2\sqrt{a\omega(a\omega - m)}}{m}\cos^2\frac{\theta}{2}\,,$$
$$({}_{+1}T_m(\omega;\theta))_2 = 1 - \frac{2a\omega - 2\sqrt{a\omega(a\omega - m)}}{m}\cos^2\frac{\theta}{2}\,.$$

In the case $m = 0$ it is easy to solve the equation directly and arrive at the following overall solutions:

$$({}_{+1}S_0(\omega;\theta))_1 = e^{-a\omega\cos\theta}\tan\frac{\theta}{2}, \qquad ({}_{+1}S_0(\omega;\theta))_2 = e^{-a\omega\cos\theta}\cot\frac{\theta}{2}\,.$$

The behavior of the solutions to TAE we found at the two singularities $\theta = 0$ and $\theta = \pi$ can be easily obtained from the expressions above. Each one is divergent either at the one or at the other singularity. The exact expressions for $m \neq 0$ are:

$$
({}_{+1}S_m(\omega;\theta))_{1,2} \sim \begin{cases} \theta^{-(m+1)} & \text{at } \theta \to 0 \ \text{ for } m \geq 1 \\ \\ (\pi-\theta)^{-(m+1)} & \text{at } \theta \to \pi \ \text{ for } m \leq 1 \end{cases} . \tag{29}
$$

For $m = 0$ we have $({}_{+1}S_0(\omega;\theta))_1 \sim \theta^{-1}$ at $\theta \to 0$ and $({}_{+1}S_0(\omega;\theta))_2 \sim (\pi-\theta)^{-1}$ at $\theta \to \pi$. If we perform the transformation $\theta \mapsto \pi - \theta$ we obtain the solutions for $s = -1$:

$$
({}_{-1}S_m(\omega;\theta))_{1,2} = \frac{e^{a\omega\cos\theta}}{\sin\theta}\left(\cot\frac{(\pi-\theta)}{2}\right)^m ({}_{-1}T_m(\omega;\theta))_{1,2}\,, \tag{30}
$$

where for $m \neq 0$:

$$
\begin{aligned}
({}_{-1}T_m(\omega;\theta))_1 &= 1 - \frac{2a\omega + 2\sqrt{a\omega(a\omega - m)}}{m}\sin^2\frac{\theta}{2}\,,\\
({}_{-1}T_m(\omega;\theta))_2 &= 1 - \frac{2a\omega - 2\sqrt{a\omega(a\omega - m)}}{m}\sin^2\frac{\theta}{2}\,,
\end{aligned}
$$

and for $m = 0$:

$$
({}_{-1}S_0(\omega;\theta))_1 = e^{a\omega\cos\theta}\cot\frac{\theta}{2}, \qquad ({}_{-1}S_0(\omega;\theta))_2 = e^{a\omega\cos\theta}\tan\frac{\theta}{2}\,.
$$

It can be shown that because of the orthogonality properties of Heun polynomials [28], the following relations hold for the functions $({}_{\pm1}S_m(\omega;x))_j$ separately for $s = 1$ and for $s = -1$:

$$
\int_0^1 ({}_{\pm1}S_m(\omega;{}_{\pm1}x))_j({}_{\pm1}S_m(\omega;{}_{\pm1}x))_l d({}_{\pm1}x) = 0 \qquad j \neq l \quad j,l = 1,2. \tag{31}
$$

The second condition (A.10) for a polynomial solution of the radial equation leads to the expressions for the separation "constant" E_m for both $s = +1$ and for $s = -1$. For any m we have [39, 40]

$$
\begin{aligned}
{}_{\pm1}E_m(a\omega)_1 &= -a^2\omega^2 + 2a\omega m + 2\sqrt{a\omega(a\omega - m)}\\
{}_{\pm1}E_m(a\omega)_2 &= -a^2\omega^2 + 2a\omega m - 2\sqrt{a\omega(a\omega - m)}\,.
\end{aligned}
$$

Surprisingly, these expressions are the same both for the TRE and TAE (see Section 3.2).

Hence, the second condition (A.10) does not produce an additional relation between E_m and ω. The main consequence from this result is that we can express the separation constant E_m as a function of ω but the (complex) frequency itself remains unconstrained. As a result we obtain a continuous spectrum in ω for the solutions of Teukolsky's angular and radial equations (5) and (6).

Thus, the surprising phenomenon of simultaneous fulfilment of the polynomial conditions for angular and radial equations with spin $|s| = 1$ is related to the existence of the continuous spectrum of Teukolsky Master Equation (3) in the specific boundary problem under consideration.

4.3 Polynomial Solutions for Perturbations with Spin $|s| = 1/2$

Again we have $\sigma = 0$ and this determines the dependence of the separation constant E from ω. The result again is [39, 40]

$$_{\pm\frac{1}{2}}E_m(a\omega) = -a^2\omega^2 + 2a\omega m - \frac{1}{4}.$$

Like in the electromagnetic case both conditions for polynomiality are the same. Thus in this case again we obtain a continuous spectrum. The solutions to TAE for $s = \pm\frac{1}{2}$, presented here for the first time, are:

$$_{\frac{1}{2}}S_m(\omega;\theta) = \left(\cot\frac{\theta}{2}\right)^m \frac{e^{-a\omega\cos\theta}}{\sqrt{\sin\theta}}\,, \tag{32}$$

$$_{-\frac{1}{2}}S_m(\omega;\theta) = \left(\tan\frac{\theta}{2}\right)^m \frac{e^{a\omega\cos\theta}}{\sqrt{\sin\theta}}\,. \tag{33}$$

These solutions satisfy the same orthogonality relations as in (31). The behaviour at $\theta = 0$ and at $\theta = \pi$ can be easily deduced from (32) and (33).

5 Nonexistence of Gravitational One-Way Waves of Continuous Spectrum

The derivation of the polynomial solutions for perturbation with spin $|s| = 2$ can be found in [45, 46]. Our analysis in the gravitational case follows the same lines as in the electromagnetic one and reproduces the results of these articles. As it was already discussed, with appropriate choice of the powers in (7) and (23) we achieve $\alpha = -3$ for both TRE and TAE. When we impose the second polynomiality condition we obtain quartic equations for the values of the separation “constant”. The important difference is that unlike in the electromagnetic and the neutrino cases, the Heun polynomial conditions are different for the angular and for the radial equations. The proof based on Taylor series expansions of the roots of quartic equations can be found in [39, 40]. This result is consistent with the observation by Teukolsky and Press [6], developed further by Chandrasekar [12], that the difference between the Starobinsky’s constant for the angular and for the radial equations in the gravitational case is equal to $(12M\omega)^2$. Thus in the gravitational case we have two independent conditions, relating E_m and ω which leads to a discrete spectrum of ω labelled by m and some additional indexes for the different solutions of the quartic equations.

6 Overall Solutions to Teukolsky's Master Equation for Spin $|s| = 1$

6.1 Polynomial in both r and $\cos\theta$, Diverging at $\theta = 0$ and $\theta = \pi$ Solutions

At this point we are ready to return to the original physical fields. For $s = 1$ and $s = -1$ we have respectively $\Psi = \varphi_0$ and $\Psi = \rho^{-2}\varphi_2$. With the solutions we found we can write:

$$((\varphi_0)_m(\omega; t, r, \theta, \phi))_{1,2} \sim \frac{e^{-a\omega\cos\theta} e^{im\phi}}{\sin\theta} \Big(\cot\frac{\theta}{2} \Big)^m$$

$$\times \frac{e^{-i\omega(r_* + t)}}{\Delta} \exp\Big(\frac{ima}{r_+ - r_-} \ln \Big| \frac{r - r_+}{r - r_-} \Big| \Big) ({}_+T_m(\omega;\theta))_{1,2} ({}_+H_m(\omega; r))_{1,2} \,,$$

and

$$((\varphi_2)_m(\omega; t, r, \theta, \phi))_{1,2} \sim \frac{e^{a\omega\cos\theta} e^{im\phi}}{\sin\theta} \Big(\tan\frac{\theta}{2} \Big)^m$$
$$\times \frac{e^{i\omega(r_* - t)}}{(r - ia\cos\theta)^2} \exp\Big(- \frac{ima}{r_+ - r_-} \ln \Big| \frac{r - r_+}{r - r_-} \Big| \Big) ({}_-T_m(\omega;\theta))_{1,2} ({}_-H_m(\omega; r))_{1,2} \,.$$

These expressions can be written in a more compact form if we introduce the Kerr coordinates with the relations:

$$\tilde{V} = t + r_*, \qquad \tilde{U} = t - r_*$$
$${}_+\tilde{\phi} = \phi + \frac{a}{r_+ - r_-} \ln \Big| \frac{r - r_+}{r - r_-} \Big| \,, \qquad {}_-\tilde{\phi} = \phi - \frac{a}{r_+ - r_-} \ln \Big| \frac{r - r_+}{r - r_-} \Big| \,.$$

Thus the final expressions for the solutions to Teukolsky Master Equation will be

$$((\varphi_0)_m(\omega; t, r, \theta, \phi))_{1,2} \sim$$
$$\sim \frac{e^{-i\omega\tilde{V}} e^{-a\omega\cos\theta}}{(r^2 - 2Mr + a^2)\sin\theta} ({}_+W)^m ({}_+T_m(\omega;\theta))_{1,2} ({}_+H_m(\omega; r))_{1,2} \qquad (34)$$

and

$$((\varphi_2)_m(\omega; t, r, \theta, \phi))_{1,2} \sim$$
$$\sim \frac{e^{-i\omega\tilde{U}} e^{a\omega\cos\theta}}{(r - ia\cos\theta)^2 \sin\theta} ({}_-W)^m ({}_-T_m(\omega;\theta))_{1,2} ({}_-H_m(\omega; r))_{1,2} \,,$$

where we have introduced the following expressions

$${}_+W = e^{i({}_+\tilde{\phi})} \cot({}_+\theta/2) \,, \qquad {}_-W = e^{i({}_-\tilde{\phi})} \cot({}_-\theta/2) \qquad (35)$$

to denote the stereographic projections of a unit sphere parameterized by angles $_+\tilde{\phi}$, $_+\theta = \theta$ for $s = 1$ and $_-\tilde{\phi}$, $_-\theta = \pi - \theta$ for $s = -1$ on the complex planes $C_{\pm W}$. The first formula in (35) describes stereographic projection from the North pole and the second one – from the South one. Note that after the transition from real variables $(_\pm\theta, _\pm\tilde{\phi})$ to the complex one $_+W$ one must introduce an additional phase factor $\exp(-is_\pm\tilde{\phi})$ in the spin-weighted spheroidal harmonics, due to the back rotation of the basis (See the paper by Goldberg et al. in [8].). In the case of spin $1/2$ the introduction of such a factor $\exp(\mp\frac{i}{2}{}_\pm\tilde{\phi})$ is equivalent to a transition in what follows from half-integer to integer values of the azimuthal number m and a replacement $m \to \pm 1/2$.

The two expressions in (34) together with their behavior at $r \to \infty$ and at the regular singularities of TRE and TAE provide us with a basis from which we can build solutions with specific boundary conditions. The basis describes wave collimated along the poles $\theta = 0$ and $\theta = \pi$. Depending on the values of the parameters ω and m, when ω and $\varpi = \omega - m\Omega_+$ have the same sign, the basis describes waves having the same direction of propagation at both r_+ and at infinity. Otherwise, when ω and $\varpi = \omega - m\Omega_+$ have opposite signs, the waves have opposite directions of propagation at r_+ and at infinity. One possible application of this basis is to try to explain the Central Engine of the Gamma Ray Bursts (GRB), discussed in [37] and [38].

6.2 Polynomial in r Regular at $\theta = 0$ and $\theta = \pi$ Solutions

We could combine the polynomial solutions to TRE with the spin-weighted spheroidal harmonics representing the regular at $\theta = 0$ and $\theta = \pi$ solution. The result will be a basis of waves which have the same properties in radial direction as those discussed above but will not be collimated along the axes of rotation. Possible application of the basis in this form is to study the influence of rotating gravitational field for formation and evolution of the Supernovae outbursts [39, 40].

7 Overall Solutions to Teukolsky Master Equation for $|s| = 1/2$, Diverging at $\theta = 0$ and $\theta = \pi$

Combining the results from (32), (33), (21), and (22) we can write down the expressions for the neutrino components χ_0 and χ_1 as:

$$\begin{aligned} (\chi_0)_m(\omega; t, r, \theta, \phi) &\sim \frac{e^{-i\omega\tilde{V}} e^{-a\omega\cos\theta}}{\sqrt{r^2 - 2Mr + a^2}\sqrt{\sin\theta}} ({}_+W)^m \,, \\ (\chi_1)_m(\omega; t, r, \theta, \phi) &\sim \frac{e^{-i\omega\tilde{U}} e^{a\omega\cos\theta}}{(r - ia\cos\theta)\sqrt{\sin\theta}} ({}_-W)^m \,. \end{aligned} \tag{36}$$

Using these solutions, which to the best of our knowledge are published for the first time, we will show how we can build regular with respect to the θ solutions. The general form of the one-way solutions, based on (36) is

$$\chi_0(\omega; t, r, \theta, \phi) = \frac{e^{-i\omega\tilde{V}} e^{-a\omega\cos\theta}}{\sqrt{r^2 - 2Mr + a^2}\sqrt{\sin\theta}} \sum_{m=-\infty}^{\infty} A_m(\omega)({}_{+}W)^m \,. \quad (37)$$

The physical model is determined by the amplitudes $A_m(\omega)$. Physically sound solutions correspond to amplitudes $A_m(\omega)$, which lead to a finite result after performing the summation. Here we consider only a formal example proving the existence of proper choice of the amplitudes giving finite results. Let us write down the part of χ_0 containing the potentially singular factor as

$$(\sin\theta)^{-1/2} \sum_{m=-\infty}^{\infty} A_m(\omega)\,({}_{+}W)^m = \\ = \sqrt{\frac{1}{2}\left(|{}_{+}W| + |{}_{+}W|^{-1}\right)} \sum_{m=-\infty}^{\infty} A_m(\omega)\,({}_{+}W)^m \;, \quad (38)$$

where $|{}_{+}W| = \cot\frac{\theta}{2} \geq 0$. At this point we proceed by choosing in an appropriate way functions $f(\omega, {}_{+}W) = \sum_{m=-\infty}^{\infty} A_m(\omega)\,({}_{+}W)^m$ thus defining the amplitudes. For example we can choose $f(\omega, {}_{+}W) = \big({}_{+}W + ({}_{+}W)^{-1} + \text{const}\big)^{-1}$ or more generally $f(\omega, {}_{+}W) = \big(P(\omega, {}_{+}W) + Q(\omega, {}_{+}W^{-1})\big)^{-1}$, where P and Q are arbitrary polynomials of degree not less than one. Then the amplitudes $A_m(\omega)$ are the coefficients in the Laurent series of the functions $f(\omega, {}_{+}W)$ with respect to ${}_{+}W$. With this choice it is clear that there are no singularities in the limits $|{}_{+}W| \to 0$ and $|{}_{+}W| \to \infty$. It can be shown that choosing the polynomials P and Q properly we preserve the collimation of the neutrino waves.

We can go one step further and formally perform the integration in (4). Thus we arrive at the following general solutions to Teukolsky's Master Equation for neutrino waves with spin weights $s = 1/2$ and $s = -1/2$ respectively:

$$\chi_0(t, r, \theta, \phi) = \frac{F_0(\tilde{V} - ia\cos\theta, {}_{+}W)}{\sqrt{\Delta}\sqrt{\sin\theta}} \,,$$

$$\chi_1(t, r, \theta, \phi) = \frac{F_1(\tilde{U} + ia\cos\theta, {}_{-}W)}{\sqrt{\sin\theta}} \,,$$

where F_0 and F_1 are arbitrary functions of their respective variables. The fact that arbitrary F_0 and F_1 satisfy Teukolsky's Master Equation can be verified directly. The exact forms of F_0 and F_1 are to be determined by the specific physical situations. The above formal examples demonstrate mathematical technics, which make possible the application of singular solutions to Teukolsky's Master Equation for description of physical reality.

8 Conclusion

In the paper we presented an approach for solving Teukolsky Master Equation based on the use of the confluent Heun equation. After separating the variables we showed that both TRE and TAE can be transformed into the non-symmetric canonical form of the confluent Heun equation. The transformation depends on a set of parameters which when properly chosen lead to polynomial solutions to both Heun equations related to TRE and to TAE. The surprising result (which simply means that there should be a deeper physical explanation we do not understand yet) is that for neutrino and for electromagnetic perturbations we find solutions which have continuous spectrum, but for gravitational perturbations this does not happen. The richness of the results we found give us the opportunity to construct different types of solutions in accordance with the specific boundary problem we want to study.

There are many possible directions we intend to pursue. One possibility is to see if indeed we can explain some of the basic features of the GRB's and of Supernovae using the basis we found. Related problem is to investigate further the stability of Kerr black holes.

Acknowledgements

The authors would like to thank Dimo Arnaudov and Denitsa Staicova for valuable comments and suggestions.

P.F. is grateful to Professor Saul Teukolsky for his comments on the problems, related with present article, and especially for raising the question about the use of singular solutions of Teukolsky's Master Equation.

This article was partially supported by the Foundation "Theoretical and Computational Physics and Astrophysics" and by the Bulgarian National Scientific Fund under contracts DO-1-895 and DO-02-136.

Author Contributions

In 2008 R.B. joined the comprehensive program for of the use of Heun's functions, developed by P.F. since 2005. He made calculations for Teukolsky radial and angular equations for electromagnetic case (i.e. for spin-weight 1) in the notations of reference [28], which simplify the polynomial conditions and discovered that these conditions coincide for both Teukolsky radial and angular equations. R.B. confirmed the results for spin-weight $1/2$, obtained previously by P.F. in Maple-notation and wrote down the overall solutions in section 7. The text of the present article was written by R.B.

P.F. developed the program for of the use of Heun's functions for solution of

Regge-Wheeler and Teukolsky equations since 2005. He found all specific basic classes of solutions to these equations, in particular, all polynomial solutions for different integer and half-integer spin-weights, as well as justification of some properties of the confluent Heun functions. P.F. explained the relation of the coinciding polynomial conditions for electromagnetic and neutrino perturbations with a novel continuous spectrum of Teukolsky master equation and discovered the method of deriving regular solutions of this equation using proper superposition of the singular polynomial solutions. He formulated the possible astrophysical applications of the obtained mathematical results. P.F. is responsible for the references, some corrections and editing of the text.

Appendix: Heun Equation and Heun Functions

In this appendix we remind the reader some basic information about the confluent Heun equation and its solutions, following the notations of the reference [28].

In subsection A.1 we give a brief description of the non-symmetrical canonical form of confluent Heun equation and its solutions. In subsection A.2 a basic information about local Frobenius and Tomé solutions around regular and irregular singular points is presented. In subsection A.3 we remind the basic information about polynomial solutions in notations of [28]. In subsection A.4 we describe the correspondence between these notations and the notations, used in the basic articles [25, 26] on the modern general theory of all kinds of Heun equations and the properties of their solutions. The last notations are used in [30–40]. At present these conventions become more popular, because they are used in the computer package Maple, based on the articles [25,26]. This package is still the only one for analytical and numerical computer calculations with Heun equations and Heun functions. In Maple's Help one can find an available and rich collection of relations and properties of Heun functions of all kinds.

A.1 Non-Symmetrical Canonical Form of Confluent Heun Equation and Its Solutions

The general Heun equation is a second order ODE of Fuchsian type with four regular singular points. In the present paper we have to solve the confluent Heun equation (CHE) for different cases. It is relatively well studied [23–29], but there still exist essential gaps in the theory. CHE can be obtained from the general Heun equation by coalescing of two of the singular points by redefining certain parameters and taking the appropriate limits. In this way two regular singular points coalesce into one irregular (in general) point. The solutions of the confluent Heun equation are relatively well-studied special functions, already included in modern computer package Maple. These functions represent non-trivial generalization of known hypergeometric functions, yet have richer properties, because confluent Heun equation has one more singular point than the hypergeometric. One of the canonical forms of the confluent Heun equation is the so called non-symmetric canonical form [28]:

$$\frac{d^2H}{dz^2} + \left(4p + \frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{dH}{dz} + \frac{4\alpha p z - \sigma}{z(z-1)}H = 0. \tag{A.1}$$

The only regular Frobenius' type solution to (A.1) about the regular singular point $z = 0$ is denoted by $Hc^{(a)}(p, \alpha, \gamma, \delta, \sigma; z)$. It is defined for non-integral values of $(1-\gamma)$ in the domain $|z| < 1$ by the condition

$$Hc^{(a)}(p, \alpha, \gamma, \delta, \sigma; 0) = 1. \tag{A.2}$$

In [28] it is called the "angular solution" of the confluent Heun equation.

Another solution, is the Tomé's type asymptotical solution $Hc^{(r)}(p, \alpha, \gamma, \delta, \sigma; z)$. It is defined for complex $p = |p|e^{i\varphi}$ in the domain $|z| > 1$ by the condition:

$$\lim_{|z|\to\infty} z^{\alpha} Hc^{(r)}(p, \alpha, \gamma, \delta, \sigma; -|z|e^{-i\varphi}) = 1. \tag{A.3}$$

In [28] it is called the "radial solution" of the confluent Heun equation.

Different pairs of local solutions can be constructed using the combinations of four known independent transformations of the parameters, which preserve the chosen canonical form of the Heun Equation. For example by interchanging the regular singular points $z_1 = 0$ and $z_2 = 1$:

$$z \mapsto 1 - z,$$

one obtains the following new solutions:

$$\begin{aligned} &Hc^{(a)}(-p, \alpha, \delta, \gamma, \sigma + 4p\alpha; 1 - z) \\ &Hc^{(r)}(-p, \alpha, \delta, \gamma, \sigma + 4p\alpha; 1 - z)\,. \end{aligned} \tag{A.4}$$

All possible sets of local solutions to Regge-Wheeler and Teukolsky equations were described for the first time in [39, 40].

A.2 Power-Series Solutions of the Confluent Heun Equation

Taylor series expansion about the regular singularity $z = 0$

If we expand the solution $Hc^{(a)}(p, \alpha, \gamma, \delta, \sigma; z)$ as a power series

$$Hc^{(a)}(p, \alpha, \gamma, \delta, \sigma; z) = \sum_{k=0}^{\infty} c_k^{(a)} z^k \tag{A.5}$$

then we get a three-term recurrence relation for the coefficients $c_k^{(a)}$:

$$
\begin{aligned}
&f_k^{(a)} c_{k+1}^{(a)} + g_k^{(a)} c_k^{(a)} + h_k^{(a)} c_{k-1}^{(a)} = 0 \\
&c_{-1} = 0, \qquad c_0 = 1\,,
\end{aligned}
\tag{A.6}
$$

where

$$
\begin{aligned}
g_k^{(a)} &= k(k - 4p + \gamma + \delta - 1) - \sigma \\
f_k^{(a)} &= -(k+1)(k+\gamma) \\
h_k^{(a)} &= 4p(k + \alpha - 1)\,.
\end{aligned}
$$

The radius of convergence of the series (A.5) is equal to unity, which is the distance to the next regular singular point [28].

Laurent series expansion about the singular point at infinity

Another power series can be constructed at infinity. In general this series is not convergent but only asymptotic. For the function $Hc^{(r)}(p, \alpha, \gamma, \delta, \sigma; z)$ we will have the expansion

$$
Hc^{(r)}(p, \alpha, \gamma, \delta, \sigma; z) = z^{-\alpha} \sum_{k=0}^{\infty} c_k^{(r)} z^{-k}. \tag{A.7}
$$

The three-term recurrence relation for the coefficients $c_k^{(r)}$ reads

$$
\begin{aligned}
&f_k^{(r)} c_{k+1}^{(r)} + g_k^{(r)} c_k^{(r)} + h_k^{(r)} c_{k-1}^{(r)} = 0 \\
&c_{-1} = 0, \qquad c_0 = 1\,,
\end{aligned}
\tag{A.8}
$$

with the following expressions[5] for $f_k^{(r)}$, $g_k^{(r)}$, and $h_k^{(r)}$:

$$
\begin{aligned}
g_k^{(r)} &= (\alpha + k)(\alpha + k + 4p - \gamma - \delta + 1) - \sigma \\
f_k^{(r)} &= -4p(k+1) \\
h_k^{(r)} &= -(k + \alpha - 1)(\alpha + k - \gamma)\,.
\end{aligned}
$$

It is easy to show that in general the series expansion (A.7) diverges [28].

A.3 Polynomial Solutions of the Confluent Heun Equation

Let us consider the case in which the parameter α has a fixed negative integral value

$$
\alpha = -N, \qquad N \in \mathbb{N}. \tag{A.9}
$$

In this case the coefficient $h_{N+1}^{(a)}$ vanishes for both expansions above. If we impose in addition the second condition that

$$
c_{N+1}^{(a)} = 0 \tag{A.10}
$$

[5] Note that these expressions are somewhat different from those in [28].

then the recurrence relation breaks down and we obtain a polynomial of N-th order instead of the infinite series. Since the coefficients $g_k^{(a)}$ are linear functions of the parameter σ the equation $c_{N+1}^{(a)} = 0$ is an algebraic equation of $(N+1)$-th order and thus it has $(N+1)$ zeros $\sigma_0, \sigma_1, \ldots, \sigma_N$ [28, 29].

In [39, 40] one can find an explicit representation of the coefficient $c_{N+1}^{(a)} = 0$ in form of a specific determinant Δ_{N+1}. This form is most convenient for practical calculations.

A.4 Correspondence between the Notations of Present Article and the Notations, Used in Computer Package Maple

The computer package Maple uses the conventions of the basic articles [25, 26]. In the Maple notation HeunC$(\alpha, \beta, \gamma, \delta, \eta, z)$ for the solution (A.2) the parameters $\alpha, \beta, \gamma, \delta, \eta$ are related in the following way with the parameters of the non-symmetrical canonical form of confluent Heun equation [28], used in the present article, too:

$$\alpha_{_{Maple}} = 4p, \;\; \beta_{_{Maple}} = \gamma - 1, \;\; \gamma_{_{Maple}} = \delta - 1, \tag{A.11}$$
$$\delta_{_{Maple}} = 4p\alpha - 2p(\gamma + \delta), \;\; \eta_{_{Maple}} = 2p\gamma - \frac{\gamma\delta - 1}{2} - \sigma.$$

In [29,39,40] a modified Maple-like parametrization of the confluent Heun equation is used:

$$\frac{d^2H}{dz^2} + \left(\alpha + \frac{\beta+1}{z} + \frac{\gamma+1}{z-1}\right)\frac{dH}{dz} + \left(\frac{\mu}{z} + \frac{\nu}{z-1}\right)H = 0. \tag{A.12}$$

The equation (A.12) has a uniform shape. This uniform parametrization simplifies the explicit expressions for the coefficients in the series (A.5) and (A.7). For the parameters μ and ν one obtains the following relations with the parametrization, used in present article:

$$\mu = \sigma, \;\; \mu + \nu = 4p\alpha.$$

The first polynomial condition (A.9) in Maple notations, as well as in the above uniform parametrization, reads:

$$\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + N + 1 = 0.$$

It yields discrete values $\delta = -\alpha\left(\frac{1}{2}(\beta+\gamma) + N + 1\right)$ of the Maple parameter δ. Hence, the name δ_N-condition [29], [39, 40].

References

[1] T. Regge, J.A. Wheeler (1957) *Phys. Rev.* **108** 1063-1069.

[2] F.J. Zerilli (1970) *Phys. Rev. Lett.* **24** 737-738.

[3] M. Johnston, R. Ruffini, F.J. Zerilli (1973) *Phys. Rev. Lett*. **31** 1317-1319.

[4] S.A. Teukolsly (1972) *Phys. Rev. Lett*. **29** 1114-1118.

[5] S.A. Teukolsly (1973) *Astrophys. J.* **185** 635-647.

[6] W.H. Press, S.A. Teukolsly (1973) *Astrophys. J.* **185** 649-674.

[7] S.A. Teukolsly, W.H. Press (1973) *Astrophys. J*. **193** 443-461.

[8] J.N. Goldberg, A.J. Makfarlane, E.T. Newman, F. Rohrlich, E.C.G. Sudarshan (1967) *J. Math. Phys*. **8** 2155-2161.

[9] S.R. Dolan, (2006) PhD Thesis,: Scattering, Absorbtion and Emission by Black Holes. Trinity Hall and Cavendish Laboratory.

[10] S. Dolan, J. Gair (2009) *Class. Quant. Grav*. **26** 175020.

[11] E. Newman, R. Penrose (1962) *J. Math. Phys*. **3** 566-578.

[12] S. Chandrasekhar (1983) *The Mathematical Theory of Black Holes*, Vol. 1, 2 Oxford University Press, Oxford.

[13] N. Andersson (1972) *Phys. Rev. D* **55** 468-479.

[14] K. Kokkotas, B. Schmidt (1999) "Quasi-Normal Modes of Stars and Black Holes. Living Reviews in Relativity", http://relativity.livingreviews.org/Articles/lrr-1999-2/ (1999). Acessed July 19 2009.

[15] H.-P. Nollert (1999) *Class. Quantum Grav*. **16** R159-R216.

[16] S. Hod (1998) *Phys. Rev. D* **58** 104022.

[17] R. Price (1972) *Phys. Rev. D* **5** 2419-2438.

[18] E.W. Leaver (1985) *Proc. R. Soc. Lond. A* **402** 285-298.

[19] G. Marcilhacy (1983) *Lett. Nuovo Cim*. **37** 300-302.

[20] J. Balandin, R. Pons, G. Marcilhacy (1938) *Lett. Nuovo Cim.* **38** 561-567.

[21] D.V. Galtsov, A.A. Ershov (1989) *Sov. Phys. J*. **32** 13-18.

[22] H. Suzuki, E. Takasugi, H. Umetsu (1998) *Prog. Theor. Phys*. **100** 491-505.

[23] K. Heun (1889) *Math. Ann*. **33** 161-179.

[24] H. Bateman, A. Erdélyi (1955) *Higher Transcendental Functions* Vol. 3. McGrow-Hill, New York Toronto London.

[25] A. Decarreau, M.C. Dumont-Lepage, P. Maroni, A. Robert, A. Ronveaux (1978) *Ann. Soc. Sci. Bruxelles* **92** I-II, 53-78.

[26] A. Decarreau, P. Maroni, A. Robert (1978) *Ann. Soc. Sci. Buxelles* **92** III, 151-189.

[27] S. Slavyanov, W. Lay (2000) *Special Functions, A Unified Theory Based on Singularities*, Oxford Univ. Press, Oxford.

[28] A. Ronveaux (ed.) (1995) Heun's Differential Equations. Oxford Univ. Press, Oxford.

[29] P.P. Fiziev (2010) *J. Phys. A: Math. Theor*. **43** 035203.

[30] P.P. Fiziev (2006) *Class. Quantum Grav*. **23** 2447-2468.

[31] P.P. Fiziev (2006) Exact Solutions of Regge-Wheeler Equation in the Schwarzschild Black Hole Interior. gr-qc/0603003.

[32] P.P. Fiziev (2007) *J. Phys. Conf. Ser*. **66** 012016.

[33] P.P. Fiziev (2007) Exact Solutions of Regge-Wheeler and Teukolsky Equations. Talk given at the seminar of the Astrophysical Group of the Uniwersytet Jagiellonski, Institut Fizyki, Cracow, Poland, http://tcpa.uni-sofia.bg/research/Cracow_23_05_07.pdf. Accessed July 18 2009.

[34] P.P. Fiziev (2007) Exact Solutions of Teukolsky Equations. Talk given at the Conference Gravity, Astrophysics and Strings at the Black Sea, Primorsko, Bulgaria, http://tcpa.uni-sofia.bg/conf/GAS/files/Plamen_Fiziev.pdf. Accessed July 18 2009.

[35] P.P. Fiziev, D.R. Staicova (2007) A new model of the Central Engine of GRB and the Cosmic Jets. Talk given at the Conference Gravity, Astrophysics and Strings at the Black Sea, Primorsko, Bulgaria, http://tcpa.uni-sofia.bg/conf/GAS/files/GRB_Central_Engine.pdf. Accessed July 18 2009.

[36] P.P. Fiziev, D.R. Staicova (2007) A new model of the Central Engine of GRB. Talk given at the Fourth Aegean Summer School, Lesvos, Greece, http://tcpa.uni-sofia.bg/research/DStaicova_Lesvos.pdf. Accessed July 18 2009.

[37] P.P. Fiziev, D.R. Staicova (2009) *Bulg. Astrophys. J*. **11** 3; arXiv:0902.2408.

[38] P.P. Fiziev, D.R. Staicova (2009) *Bulg. Astrophys. J*. **11** 13; arXiv:0902.2411.

[39] P.P. Fiziev (2009) Classes of Exact Solutions to Regge-Wheeler and Teukolsky Equations. gr-qc/0902.1277.

[40] P.P. Fiziev (2010) *Class. Quantum Grav*. **27** 135001; gr-qc/0908.4234.

[41] E.L. Ince (1956) *Ordinary Differential Equations*, Dover Publications, New York.

[42] C.W. Misner, K.S. Thorne, J.A. Wheeler (1973) *Gravitation*, Freeman, San Francisco.

[43] E.W. Leaver (1986) *Phys. Rev. D* **34** 384-408.

[44] E.W. Leaver (1990) *Phys. Rev. D* **41** 2986-2997.

[45] F.R.S. Chandrasekhar (1984) *Proc. Roy. Soc. London A* **392** 1-13.

[46] A.M. Van den Brink (2000) *Phys. Rev. D* **62** 064009.

[47] E. Fackerell, E.R. Grossman (1977) *J. Math. Phys*. **18** 1849-1854.

[48] E.W. Leaver (1986) *J. Math. Phys*. **27** 1238-1265.

[49] E. Seidel (1989) *Clas. Quant. Grav*. **6** 1057-1062.

[50] Casals, M., Ottewill, A.C.: High frequency asymptotics for the spin-weighted spheroidal equation. Phys. Rev. D 71, 064025 (2005)

[51] E. Berti, V. Cardoso, M. Casals (2006) *Phys. Rev. D* **73** 024013.

[52] M. Abramowitz, I.A. Stegun (1965) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York.