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Bianchi Type V Cosmological Model in Scale Invariant Theory

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Abstract. In this paper, an attempt has been taken to study the perfect fluid distribution in the scale invariant theory of gravitation, when the space-time described by Bianchi type V metric with a time dependent gauge function. In this theory, a non-singular steady state model of the universe is constructed and some physical behaviors of the model are studied.

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1 Introduction

Wesson (1980) has formulated a scale invariant theory of gravitation using a gauge function $\beta(x^i)$, where x^i, $i = 1, 2, 3, 4$ are coordinates in the four-dimensional space-time and the tensor field is identified with the metric tensor g_{ij}. This theory is both coordinate and scale invariant in nature. The field equations formulated by Wesson (1981) for the combined scalar and tensor fields are

$$G_{ij} + 2\frac{\beta_{;ij}}{\beta} - 4\frac{\beta_{,i}\beta_{,j}}{\beta^2} + \left(g^{ab}\frac{\beta_{,a}\beta_{,b}}{\beta^2} - 2g^{ab}\frac{\beta_{;ab}}{\beta}\right) g_{ij} + \Lambda_0\beta^2 g_{ij} = -\kappa T_{ij} \quad (1)$$

with

$$G_{ij} \equiv R_{ij} - \frac{1}{2}R g_{ij}\,. \quad (2)$$

Here, G_{ij} is the conventional Einstein tensor involving g_{ij}. Semicolon and comma respectively denote covariant differentiation with respect to g_{ij} and partial differentiation with respect to coordinates. The cosmological term Λg_{ij} of Einstein theory is transformed to $\Lambda_0\beta^2 g_{ij}$ in scale invariant theory with a dimensionless constantΛ_0. T_{ij} is the energy momentum tensor of the matter field and $\kappa = 8\pi G/c^4$.

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Omote (1971,1974), Dirac (1973,1974), Hoyle and Narlikar (1974) and Canuto *et al.* (1977) are some of the authors who have tried to establish a new scale invariant theory in order to have a good match of theoretical results with the experimental ones. However, Wesson's (1981) formulation of scale invariant theory is so far the best one to describe all the gravitational interactions in scale free manner. There has been considerable interest in spatially homogeneous and anisotropic Bianchi cosmological models. The existence of such models allows a theoretical discussion of many important aspects. (Ryan and Shapely, 1975).

In this theory Mohanty and Daud (1997) have constructed the vacuum cosmological models when the space-time described by homogeneous and anisotropic Bianchi type I metric with different types of Dirac gauge functions. It is shown there that the models reduce to Kasner (1921) model when dimensionless cosmological constant $\Lambda_0 = 0$, but $\Lambda_0 \neq 0$, the model isotropize as in Einstein theory.

Recently, Mohanty and Mishra (2003) have studied the feasibility of Bianchi type VIII and IX space-times with a time dependent gauge function and a matter field in the form of perfect fluid. In that paper, they have constructed a radiating model of the universe for the feasible Bianchi type VIII space-time. Moreover Mishra (2004) has constructed the non-static plane symmetric Zeldovich fluid model in this theory with a time dependent gauge function.

In this paper, we develop the Bianchi type V cosmological model in Wesson's scale invariant theory of gravitation when the source of the matter field is that of a perfect fluid. The field equations of scale invariant theory are set up in Section 2. In Section 3, we obtain the explicit exact solution for the equation of state $p = \rho c^2/3$. Some physical properties of the model are discussed in Section 4 and finally in Section 5 concluding remarks of the findings are given.

2 Field Equations

We consider here the Bianchi type V metric with a gauge function $\beta = \beta(ct)$ in the form

$$ds_W^2 = \beta^2 ds_E^2 \tag{3}$$

with

$$ds_E^2 = -c^2dt^2 + e^{2A}dx^2 + e^{2x+2B}dy^2 + e^{2x+2C}dz^2, \tag{4}$$

where A, B and C are functions of t only, and c is the velocity of light.

Here we intend to take an attempt to build cosmological models in this space-time with a perfect fluid having the energy momentum tensor of the form

$$T_{ij}^m = \left(p_m + \rho_m c^2\right) U_i U_j + p_m g_{ij}, \tag{5}$$

together with

$$g_{ij}U^iU^j = -1,$$

where U^i is the four-velocity vector of the fluid. p_m and ρ_m are the proper isotropic pressure and energy density of the matter, respectively.

The non-vanishing components of conventional Einstein's tensor (2) for the metric (4) are

$$G_{11} = \frac{e^{2A}}{c^2}\left[B_{44} + C_{44} + B_4^2 + C_4^2 + B_4C_4 - c^2e^{-2A}\right] \tag{6}$$

$$G_{14} = -2A_4 + B_4 + C_4 \tag{7}$$

$$G_{22} = \frac{e^{2B+2x}}{c^2}\left[A_{44} + C_{44} + A_4^2 + C_4^2 + A_4C_4 - c^2e^{-2A}\right] \tag{8}$$

$$G_{33} = \frac{e^{2C+2x}}{c^2}\left[A_{44} + B_{44} + A_4^2 + B_4^2 + A_4B_4 - c^2e^{-2A}\right] \tag{9}$$

$$G_{44} = -\left[A_4B_4 + B_4C_4 + A_4C_4 - 3c^2e^{-2A}\right]. \tag{10}$$

Here afterwards the suffix 4 after a field variable denotes exact differentiation with respect to time t only.

Using the comoving coordinate frame, where $U^i = \delta^i_4$, the non-vanishing components of the field Eq. (1) for the metric (3) can be written in the following explicit form:

$$G_{11} = -\kappa p_m e^{2A} - \frac{e^{2A}}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(B_4 + C_4) + \Lambda_0\beta^2c^2\right] \tag{11}$$

$$G_{14} = 0, \quad i.e. \quad 2A = B + C + k_1,\ k_1 \text{ is an integrating constant} \tag{12}$$

$$G_{22} = -\kappa p_m e^{2B+2x} - \frac{e^{2B+2x}}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(A_4 + C_4) + \Lambda_0\beta^2c^2\right] \tag{13}$$

$$G_{33} = -\kappa p_m e^{2C+2x} - \frac{e^{2C+2x}}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(A_4 + B_4) + \Lambda_0\beta^2c^2\right] \tag{14}$$

$$G_{44} = -\kappa\rho_m c^4 + \left[3\frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(A_4 + B_4 + C_4) + \Lambda_0\beta^2c^2\right]. \tag{15}$$

Now, Eq. (1) and Eqs. (11)–(14) (Wesson [1980,1981]) suggest the definitions of quantities p_v (vacuum pressure) and ρ_v (vacuum density) which involve neither the Einstein tensor of conventional theory nor the properties of conventional matter. These two quantities can be obtained as

$$2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(2A_4) + \Lambda_0\beta^2c^2 = \kappa p_v c^2 \tag{16}$$

$$2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(3A_4 - B_4) + \Lambda_0\beta^2c^2 = \kappa p_v c^2 \tag{17}$$

$$2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(3A_4 - B_4) + \Lambda_0\beta^2 c^2 = \kappa p_v c^2 \tag{18}$$

$$3\frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}(3A_4) + \Lambda_0\beta^2 c^2 = -\kappa\rho_v c^4. \tag{19}$$

It is evident from Eqs. (16)–(18) that p_v is unique only when

$$A = k_2 B, \qquad (\text{since}\beta_4 \neq 0), \tag{20}$$

where k_2 is an integrating constant.

Using Eq. (20) in Eqs. (16)–(19), the pressure and energy density for vacuum case can be obtained as

$$p_v = \frac{1}{\kappa c^2}\left[2\frac{\beta_4}{\beta}(2A_4) + 2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + \Lambda_0\beta^2 c^2\right] \tag{21}$$

$$\rho_v = -\frac{1}{\kappa c^4}\left[2\frac{\beta_4}{\beta}(3A_4) + 3\frac{\beta_4^2}{\beta^2} + \Lambda_0\beta^2 c^2\right]. \tag{22}$$

Here p_v and ρ_v relate to the properties of vacuum only in conventional physics. The definition of the above quantities is natural as regards to the scale invariant properties of the vacuum. The total pressure and energy density can be defined as

$$p_t \equiv p_m + p_v \tag{23}$$

$$\rho_t \equiv \rho_m + \rho_v \tag{24}$$

Using the aforesaid definitions of p_t and ρ_t, the field equations in scale invariant theory, *i.e.* (11)–(15) can now be written by using the components of Einstein tensor (6)–(10) and the results obtained in Eqs. (20)–(22) as

$$2A_{44} + 3A_4^2 - c^2 e^{-2A} = -\kappa p_t c^2 \tag{25}$$

$$3A_4^2 - 3c^2 e^{-2A} = \kappa\rho_t c^4. \tag{26}$$

3 Solution

Eqs. (25)–(26) are two equations in three unknowns A, p_t and ρ_t. For the complete determinacy one extra condition is needed. So we take the equation of state

$$p_t = \frac{\rho_t c^2}{3}, \tag{27}$$

with the help of which, Eqs. (25) and (26) yield

$$A = \log(c^2t^2 + 2k_3t + k_4)^{\frac{1}{2}}, \tag{28}$$

where k_3 and k_4 are integrating constants.

Without loss of generality, let us assume that $k_1 = 1$ in Eq. (12) and $k_2 = 1$ in Eq. (20), then we have

$$A = B = C = \log(c^2t^2 + 2k_3t + k_4)^{\frac{1}{2}} . \tag{29}$$

Now the total pressure 'p_t' and energy density 'ρ_t' can be given as

$$p_t = \frac{\rho_t c^2}{3} = \frac{1}{\kappa c^2}\left[\frac{\left(k_3^2 - k_4c^2\right)}{\left(c^2t^2 + 2k_3t + k_4\right)^2}\right] . \tag{30}$$

The pressure and energy density corresponding to vacuum case can be calculated as

$$p_v = -\frac{1}{\kappa c^2}\left[\frac{4\left(c^2t + k_3\right)}{t\left(c^2t^2 + 2k_3t + k_4\right)} - \frac{\Lambda_0 + 3}{t^2}\right] \tag{31}$$

$$\rho_v = \frac{1}{\kappa c^4}\left[\frac{6\left(c^2t + k_3\right)}{t\left(c^2t^2 + 2k_3t + k_4\right)} - \frac{\Lambda_0 + 3}{t^2}\right] . \tag{32}$$

Now the matter pressure and density can be obtained as

$$p_m = \frac{1}{\kappa c^2}\left[\frac{\left(k_3^2 - k_4c^2\right)}{\left(c^2t^2 + 2k_3t + k_4\right)^2} + \frac{4\left(c^2t + k_3\right)}{t\left(c^2t^2 + 2k_3t + k_4\right)} - \frac{\Lambda_0 + 3}{t^2}\right] \tag{33}$$

$$\rho_m = \frac{1}{\kappa c^4}\left[\frac{3\left(k_3^2 - k_4c^2\right)}{\left(c^2t^2 + 2k_3t + k_4\right)^2} - \frac{6\left(c^2t + k_3\right)}{t\left(c^2t^2 + 2k_3t + k_4\right)} + \frac{\Lambda_0 + 3}{t^2}\right] \tag{34}$$

So, the radiating Bianchi type V model in scale invariant theory is given by the Eqs. (29) and (30) and the metric in this case is

$$ds_w^2 = \frac{1}{c^2t^2}\left[-c^2dt^2 + \left(c^2t^2 + 2k_3t + k_4\right)\left\{dx^2 + e^{2x}\left(dy^2 + dz^2\right)\right\}\right] . \tag{35}$$

With the help of suitable time coordinate transformation the metric can be put in the form

$$ds_w^2 = -dT^2 + Q^2(T)\left[dx^2 + e^{2x}\left(dy^2 + dz^2\right)\right] , \tag{36}$$

where

$$Q(T) = \left(1 + \frac{2k_3}{c^2}e^{-T} + \frac{k_4}{c^2}e^{-2T}\right)^{\frac{1}{2}} .$$

4 Some Physical Properties of the Model

The scalar expansion,$\theta = U^i_{;i} = 3Q_T/Q$ for the model given by Eq. (36) takes the form

$$\theta = -3\left(\frac{k_3 e^{-T} + k_4 e^{-2T}}{c^2 + 2k_3 e^{-T} + k_4 e^{-2T}}\right). \tag{37}$$

Thus, we find

$$\theta \to -3\left(\frac{k_3 + k_4}{c^2 + 2k_3 + k_4}\right)$$

as $T \to 0$ and $\theta \to 0$ as $T \to \infty$.

If $k_3, k_4 < 0$, the model represents expanding one for $T < T_1$

$$\left(= \ln \frac{k_4}{-k_3 - \sqrt{k_3^2 - k_4 c^2}}\right)$$

and the rate of expansion is accelerated when $T > T_2$

$$\left(= \ln \frac{k_3 k_4}{-k_4 c^2 + \sqrt{k_4^2 c^4 - k_3^2 k_4 c^2}}\right)$$

and beyond T_2, the rate of expansion is decelerated.

It is also observed that

$$\frac{\rho_m}{\theta^2} \to \text{constant as } T \to 0,$$

which confirms the homogeneity nature of the space-time.

The shear scalar $\sigma = 0$, which indicates that the shape of the universe remains unchanged during the evolution. Also the space-time is isotropized during evolution in scale invariant theory. The vorticity 'w' vanishes, which indicates that u^i is hypersurface orthogonal. As the acceleration u_i found to be zero, the matter particle follows geodesic path in this theory.

The spatial volume for the model (36) found to be

$$V = \left(1 + \frac{2k_3}{c^2} e^{-T} + \frac{k_4}{c^2} e^{-2T}\right)^{\frac{3}{2}} \tag{38}$$

Thus

$$V \to \left(1 + \frac{2k_3}{c^2} + \frac{k_4}{c^2}\right)^{\frac{3}{2}}$$

as $T \to 0$ and $V \to 1$ as $T \to \infty$.

So, the universe with constant volume, *i.e.*

$$\left(1+\frac{2k_3}{c^2}+\frac{k_4}{c^2}\right)^{\frac{3}{2}}$$

at initial epoch starts expanding with uniform rate till infinite future, where $V = 1$.

Further, we have $\rho_m \rightarrow$ constant as $T \rightarrow 0$ and $\rho_m \rightarrow 0$ as $T \rightarrow \infty$. Also when $T < 0$, $\rho_m =$ constant. It is interesting to note that the model is free from singularity.

Corresponding to the metric (36), the value of the Hubble parameter H is

$$H = \frac{Q_T}{Q} = -\left(\frac{k_3 e^{-T} + k_4 e^{-2T}}{c^2 + 2k_3 e^{-T} + k_4 e^{-2T}}\right), \tag{39}$$

which determines the present rate of expansion of the universe. However $H \rightarrow$ constant as $T \rightarrow 0$ and $H \rightarrow 0$ as $T \rightarrow \infty$, which indicates that the rate of expansion is accelerated for $k_3, k_4 < 0$ and decelerated for $k_3, k_4 > 0$.

Moreover for the model (36), the deceleration parameter $q = -Q_{TT}Q/Q_T^2$ becomes

$$q = 1 - \frac{\left(c^2 + 2k_3 e^{-T} + k_4 e^{-2T}\right)\left(k_3 e^{-T} + 2k_4 e^{-2T}\right)}{\left(k_3 e^{-T} + k_4 e^{-2T}\right)^2}\,. \tag{40}$$

Since $q \rightarrow$ constant $(= M,$ say$)$ as $T \rightarrow 0$, it indicates that the rate of expansion is speeding up for $M > 0$ and slowing down for $M < 0$ and q blows up at infinite future.

For the period when

$$\frac{\left(c^2 + 2k_3 e^{-T} + k_4 e^{-2T}\right)\left(k_3 e^{-T} + 2k_4 e^{-2T}\right)}{\left(k_3 e^{-T} + k_4 e^{-2T}\right)^2} > 1,$$

the model leads to inflationary one.

5 Conclusion

In conclusion, a non-singular Bianchi type V cosmological model constructed here starts evolving at initial epoch with a constant volume and ends at infinite future with unity volume and the universe expands through out the evolution for $k_3, k_4 < 0$. As far as matter is concerned the model does not admit either big bang or big crunch during evolution till infinite future. Moreover it is interesting to note that the model is Minkowskian with unity volume at both initial epoch and infinite future for $k_3 = k_4 = 0$ and the total pressure (p_t) and total density (ρ_t) vanish. It is also observed that the matter density (ρ_m) vanishes for $\Lambda_0 = 3$, but $p_m \neq 0$ for $k_3 = k_4 = 0$. This leads to unphysical situation. Thus, for a viable physical situation $\Lambda_0 \neq 3$. The model appears to be a steady state.

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