

STATISTICAL ENTROPY OF AXIAL SYMMETRY EINSTEIN-MAXWELL-DILATON-AXION BLACK HOLE

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Abstract. By using the method of quantum statistics, we directly derive the partition function of bosonic and fermionic field in axial symmetry Einstein-Maxwell-Dilaton-Axion black hole. Then via the improved brick-wall method, membrane model obtains that if we choose proper parameter, the entropy of black hole is proportional to the area of horizon. In our result, the stripped term and the divergent logarithmic term in the original brick-wall method no longer exist. In the whole process, we do not take any approximation. We offer a new simple and direct way of calculating the entropy of different complicated black holes.

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1. Introduction

Thermodynamics and statistical mechanics of black hole are the most stimulating and fast developed fields. We already know that black hole has the similar property of thermodynamic system. Based on this analogy, black hole should have entropy $S = \frac{1}{4}A_+$ (we take $C = \hbar = K_B = G = 1$), where A_+ is the area of horizon. The above expression S is called Bekenstein-Hawking entropy. Black hole has temperature $T_+ = \frac{\kappa}{2\pi}$, where κ is the acceleration of surface gravitation of horizon [1-3].

In the general relativity of Einstein, black hole's entropy is a simple geometrical quantity. Thus we can derive the laws of thermodynamics only by using classical field equations of Einstein and differential geometry. If we compare black hole with the ordinary thermodynamic system, we can easily find an important difference: black hole is an empty strong gravitation field, but ordinary

object consists of atoms and molecules. The microscopic structure of ordinary object makes it possible for people to interpret the thermodynamical property of object by using the statistical mechanics of microscopic components. However, whether black hole has the inner degree of freedom corresponding to its entropy or not is the key problem in black hole physics. In recent years, many methods have been used to study black hole's entropy [4–9]. Among all the methods, the most frequently used one is the brick-wall method advanced by G't Hooft [7]. This method is used to study the statistical property of scalar field in various black holes [10–13]. It is found that the generic expression of black hole's entropy consists of the term which is proportional to the area of its horizon plus the term which is not proportional to the area of its horizon and logarithmic divergent. However, it is doubted that firstly why the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole; secondly the state density near the event horizon is divergent; thirdly the logarithmic term is left out and L^3 is considered as the contribution of distant vacuum surrounding the system; fourthly the wave function of scalar or Dirac field is solved approximately. The above mentioned problems in the original brick-wall method are unnatural and insurmountable.

We derive the bosonic and fermionic partition functions in axial symmetry Einstein–Maxwell–Dilaton–Axion black hole directly by using the quantum statistical method [14] and then obtain the integral expression of the system's entropy. Then we use the membrane model to calculate entropy [14, 15]. As a result, the left out term and the divergent logarithmic term in original brick-wall method don't exist any more. The doubt that why the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole and the problem that the state density near the event horizon is divergent do not exist any more. We also consider the influence of spinning degeneracy of particles on the entropy. In the whole process, we avoid the difficulty in solving wave equation. Physical idea is clear, calculation is simple and the result is reasonable. It offers a neat way of studying the entropy of various complicated black holes.

2. Axial Symmetry Einstein–Maxwell–Dilaton–Axion Black Hole

The space–time linear element of axial symmetry Einstein–Maxwell–Dilaton–Axion black hole is given by [16]:

$$\begin{aligned}
 ds^2 = & -\frac{\Sigma - a^2 \sin^2 \theta}{\Delta} dt^2 + \frac{\Delta}{\Sigma} dr^2 + \Delta d\theta^2 + \frac{\sin^2 \theta}{\Delta} [(r^2 + a^2 - 2Dr)^2 \\
 & - \Sigma a^2 \sin^2 \theta] d\varphi^2 - \frac{2a \sin^2 \theta}{\Delta} [(r^2 + a^2 - 2Dr) - \Sigma] dt d\varphi
 \end{aligned} \quad (1)$$

where

$$\Sigma = r^2 - 2mr + a^2, \quad \Delta = r^2 - 2Dr + a^2 \cos^2 \theta$$

and

$$\begin{aligned} \exp(2\phi) &= \frac{W}{\Delta} = \frac{\omega}{\Delta}(r^2 + a^2 \cos^2 \theta), \quad \omega = \exp(2\phi_0), \\ K_\alpha &= K_0 + \frac{2aD \cos \theta}{W}, \\ A_t &= \frac{1}{\Delta}(Qr - ga \cos \theta), \quad A_r = A_\theta = 0, \\ A_\varphi &= \frac{1}{a\Delta}[-Qra^2 \sin^2 \theta + g(r^2 + a^2)a \cos \theta] \end{aligned}$$

where

$$M = m - D, \quad J = a(m - D), \quad Q = \sqrt{2\omega D(D - m)}, \quad P = g,$$

are mass, angular momentum, electric charge and magnetic charge of black hole respectively. The radiation temperature of black hole is as follows:

$$T_+ = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2 - 2Dr_+)} \quad (2)$$

where

$$r_\pm = \left(M - \frac{Q^2}{2\omega M} \right) \pm \sqrt{\left(M - \frac{Q^2}{2\omega M} \right)^2 - a^2}$$

are the locations of outer and inner horizons respectively

$$A_+ = 4\pi(r_+^2 + a^2 - 2Dr_+). \quad (3)$$

3. Bosonic Entropy

Based on the theory of general relativity, an observer at rest at an infinite distance gets the frequency shift of the particles from the surface of a star as follows:

$$\nu = \nu_0 \sqrt{-g'_{tt}} \quad (4)$$

where ν_0 is the natural frequency of the atoms on the surface of star and ν is the one obtained by the observer at rest at an infinite distance.

The natural radiation temperature [17, 18] got by the observer at rest at an infinite distance is as follows:

$$T = \frac{T_+}{\sqrt{-g'_{tt}}} \quad (5)$$

where T_+ is the equilibrium temperature and

$$\begin{aligned} g'_{tt} &= \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} \\ &= -\frac{(r-r_+)(r-r_-)(r^2 - 2Dr + a^2 \cos^2 \theta)}{(r^2 + a^2 - 2Dr)^2 - (r-r_+)(r-r_-)a^2 \sin^2 \theta}. \end{aligned} \quad (6)$$

For bosonic gas, we calculate the partition function as follows:

$$\ln Z = -\sum_i g_i \ln(1 - e^{-\beta\epsilon_i}). \quad (7)$$

In unit volume, the number of quantum states with the energy between ϵ and $\epsilon + d\epsilon$ or the frequency between ν and $\nu + d\nu$ is as follows:

$$g(\nu) d\nu = j4\pi\nu^2 d\nu \quad (8)$$

where j is the spinning degeneracy of particles. Since in the space-time (1), the area of two-dimensional curved surface at random point r is

$$A(r) = \int dA = \int \sqrt{g} d\theta d\varphi \quad (9)$$

where $g = \begin{vmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = g_{\theta\theta}g_{\varphi\varphi}$. The volume of the lamella at random point r outside the horizon is as follows,

$$dV = A(r)\sqrt{g_{rr}} dr. \quad (10)$$

So, the partition function of the system at the lamella with random thickness at point r outside the horizon is as follows:

$$\begin{aligned} \ln Z &= \int A(r)\sqrt{g_{rr}} dr \sum_i g_i \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\epsilon_i} \\ &= j4\pi \int A(r)\sqrt{g_{rr}} dr \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-nh\nu/T} \nu^2 d\nu \\ &= j \frac{\pi^2}{90} \int \frac{A(r)\sqrt{g_{rr}}}{\beta^3} dr = j \frac{\pi^2}{90} \iiint \frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}}{\beta^3} dr d\theta d\varphi \end{aligned} \quad (11)$$

where $\frac{1}{\beta} = T$. Using the relation between entropy and partition function

$$S = \ln Z - \beta_0 \frac{\partial \ln Z}{\partial \beta_0}, \quad (12)$$

we have

$$\begin{aligned} S_b &= j \frac{2}{45} \pi^2 \frac{1}{\beta_0^3} \int \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}}}{(-g'_{tt})^{3/2}} dr \\ &= j \frac{2\pi^2}{45\beta_0^3} \int d\theta d\varphi \int \frac{[(r^2 + a^2)^2 - (r - r_+)(r - r_-)a^2 \sin^2 \theta]^2}{(r - r_+)^2 (r - r_-)^2 (r^2 + a^2 \cos^2 \theta)} \sin \theta dr \end{aligned} \quad (13)$$

where $\beta_0 = \frac{1}{T_+} = \frac{4\pi(r_+^2 + a^2 - 2Dr_+)}{r_+ - r_-}$, and $\beta = \beta_0 \sqrt{-g'_{tt}}$. In the above integral, we take for r the integral region $[r_+ + \xi, r_+ + N\xi]$ [14], where ξ is a small nonnegative quantity, N is a constant larger than one. So we have

$$\begin{aligned} S_b &= j \frac{4\pi^3}{45\beta_0^3} \int_0^\pi d\theta \int_{r_+ + \xi}^{r_+ + N\xi} \frac{[(r^2 + a^2 - 2Dr)^2 - (r - r_+)(r - r_-)a^2 \sin^2 \theta]^2}{(r - r_+)^2 (r - r_-)^2 (r^2 - 2Dr + a^2 \cos^2 \theta)} \\ &\quad \times \sin \theta dr \\ &= j \frac{4\pi^3}{45\beta_0} \int_0^\pi \frac{(r_+^2 + a^2 - 2Dr_+)^4}{(r_+ - r_-)^2 (r_+^2 - 2Dr_+ + a^2 \cos^2 \theta)} \left[\frac{N - 1}{N\xi} \right] \sin \theta d\theta \\ &\quad + G(r_+, N, \xi) \\ &= j \frac{8\pi^3}{45\beta_0^3} \frac{(r_+^2 + a^2 - 2Dr_+)^3}{(r_+ - r_-)^2} \frac{\alpha}{\sin \alpha \cos \alpha} \left[\frac{N - 1}{N\xi} \right] + G(r_+, N, \xi) \end{aligned} \quad (14)$$

where

$$\begin{aligned} G(r_+, N, \xi) &= j \frac{4\pi^3}{45\beta_0^3} \int_0^\pi \left[\frac{8(r_+ - D)(r_+^2 + a^2 - 2Dr_+)^3}{(r_+ - r_-)^2 (r_+^2 - 2Dr_+ + a^2 \cos^2 \theta)} \right. \\ &\quad - \frac{2(r_+ - D)(r_+^2 + a^2 - 2Dr_+)^4}{(r_+ - r_-)^2 (r_+^2 - 2Dr_+ + a^2 \cos^2 \theta)^2} \\ &\quad - \frac{2(r_+^2 + a^2 - 2Dr_+)^4}{(r_+ - r_-)^3 (r_+^2 - 2Dr_+ + a^2 \cos^2 \theta)} \\ &\quad \left. - \frac{2(r_+^2 + a^2 - 2Dr_+)^2 a^2 \sin^2 \theta}{(r_+ - r_-)(r_+^2 - 2Dr_+ + a^2 \cos^2 \theta)} \right] \sin \theta d\theta \ln N \end{aligned} \quad (15)$$

$$+ j \frac{4\pi^3}{45\beta_0^3} \int_0^\pi \sin \theta \, d\theta \int_{r_++\xi}^{r_++N\xi} F(r) \, dr ,$$

$$F(r) = \sum_{n=2}^{\infty} \frac{f_1^{(n)}(r_+)}{n!} (r - r_+)^{n-2} - \sum_{n=1}^{\infty} \frac{f_2^{(n)}(r_+)}{n!} (r - r_+)^{n-1} + f_3(r) , \quad (16)$$

$$f_1(r) = \frac{(r^2 + a^2 - 2Dr)^4}{(r - r_+)^2 (r^2 - 2Dr + a^2 \cos^2 \theta)} ,$$

$$f_2(r) = \frac{2(r^2 + a^2 - 2Dr)^2 a^2 \sin^2 \theta}{(r - r_+) (r^2 - 2Dr + a^2 \cos^2 \theta)} ,$$

$$f_3(r) = \frac{a^4 \sin^4 \theta}{r^2 - 2Dr + a^2 \cos^2 \theta} ,$$

$$f_1^{(n)}(r) = \frac{df_1^n}{dnr} , \quad f_2^{(n)}(r) = \frac{df_2^n}{dnr} , \quad \alpha = \arctan \frac{a}{\sqrt{r_+^2 - 2Dr_+}} . \quad (17)$$

From (3.17) in the Ref. [7], we know when $N\xi = L \gg r_+$, and if we take

$$\xi = \frac{T_+}{90} ,$$

we obtain that the entropy of black hole is proportional to the area of its horizon.

In order to let the calculated entropy be independent of the parameters N and ξ introduced in (14), we take

$$\xi = \frac{T_+}{90} \frac{\alpha}{\sin \alpha \cos \alpha} \frac{N-1}{N} , \quad (18)$$

we have

$$S_b = j\pi(r_+^2 + a^2 - 2Dr_+) + G(r_+, N, \xi) = j\frac{1}{4}A_+ + G(r_+, N, \xi) . \quad (19)$$

As $N \rightarrow 1$, $\xi \rightarrow 0$ and $N\xi \rightarrow 0$, that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole, but the black hole's entropy is as follows:

$$S_b = j\frac{1}{4}A_+ . \quad (20)$$

In our calculation, we make use of

$$\lim_{N \rightarrow 1} G(r_+, N, \xi) \rightarrow 0$$

where $A_+ = 4\pi(r_+^2 + a^2 - 2Dr_+)$ is the area of outer horizon. Since we let the integral region tend to the outer horizon, the entropy obtained in (20) is

independent of the radiation field outside horizon. It only has the property of two-dimensional membrane in three-dimensional space. So the obtained entropy has the property of two-dimensional membrane. The existence of horizon is the basic property of black hole. It has already been proved that the general existence of horizon leads to the Hawking effect [19]. And whether there is black hole's entropy or not directly involves the existence of horizon [20]. So the entropy in (20) should be black hole's entropy. When the spinning quantum number of radiation particles $j = 1$, we obtain that black hole's entropy is a quarter of the area of horizon.

4. Fermionic Entropy

For Fermionic gas, the grand partition function is as follows:

$$\ln Z = - \sum_i g_i \ln(1 + e^{-\beta\epsilon_i}). \quad (21)$$

From (8), we obtain

$$\begin{aligned} \ln Z &= \int A(r) \sqrt{g_{rr}} dr \sum_i g_i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-n\beta\epsilon_i} \\ &= \omega 4\pi \int A(r) \sqrt{g_{rr}} dr \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^{\infty} e^{-nh\nu/T} \nu^2 d\nu \\ &= \omega \frac{\pi^2}{90} \frac{7}{8} \int \frac{A(r) \sqrt{g_{rr}} dr}{\beta^3} = \omega \frac{\pi^2}{90} \frac{7}{8} \iiint \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}}}{\beta^3} dr d\theta d\varphi. \end{aligned} \quad (22)$$

Using the result of part three, we can get the fermionic entropy as follows:

$$S_f = \omega \frac{7}{8} \frac{1}{4} A_+ \quad (23)$$

where ω is the spinning degeneracy of fermionic particles.

5. Conclusion

In the above analysis, the partition functions of bosonic and fermionic field in Einstein-Maxwell-Dilaton-Axion black hole with axial symmetry are derived directly by quantum statistical method. The integral expressions of the entropy obtained in this paper are inconsistent with those obtained by WKB approximation method in the original brick-wall method. We avoid the difficulty in solving wave equation. Since we use the improved brick-wall method, membrane model, to calculate the entropy of various fields, the problem that the state density is divergent around horizon does not exist any more. By taking the value of N to make sure that the black hole and radiation field are

in the state of stable equilibrium, we overcome the shortcoming of non-stable equilibrium. In our calculation, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole. From (20) and (23) we know that the calculated entropy has nothing to do with the radiation field outside black hole and the left out term and the divergent logarithmic term in the original brick-wall method no longer exist. The obtained entropy is proportional to the area of black hole's horizon, so it can be taken as black hole's entropy.

In above analysis, we know that by using the statistical and membrane model methods, the doubt that why the entropy of the scalar or Dirac field outside the event horizon is the entropy of black hole in the original brick-wall method does not exist and the complicated approximations in solution are avoided. In the whole process, the physical idea is clear, the calculation is simple and the result is reasonable. We also consider the influence of the spinning degeneracy of particles on the entropy. For calculating the entropy in various space-time, we only need to change the red-shift factor, but the others are the same. Especially for complicated space-time, we can directly derive the entropy of various quantum particles without solving the complicated wave equation. We offer a new neat way of studying the entropy of different kinds of complicated black holes.

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