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Spin Curvatures in General Relativity and Two-Component Gravitons with Arbitrary Sources

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Abstract. We present a short review of the spin curvatures that arise within the framework of one of the Infeld-van der Waerden formalisms for general relativity. All the spinor symmetries borne by the inner structure of general relativity are thus exhibited in an explicit way. The sourceful extension formulated in an earlier work of the two-component theory of gravitons in vacuum is concisely reviewed in connection with the establishment of an important total-symmetry property related to the energy-momentum conservation law of general relativity.

KEY WORDS: ε-formalism; gravitons; sources; wave equations.

1 Introduction

The so-called $\gamma\varepsilon$-formalisms of Infeld and van der Waerden [1–4] constitute the traditional two-component spinor framework for general relativity. One of the basic procedures lying behind the construction of such formalisms involves setting up two pairs of conjugate spin spaces at every non-singular point of a curved spacetime that is endowed with a torsionless covariant derivative operator. Remarkably enough, this construction was carried out much earlier than the establishment of the definitive conditions for a curved spacetime to admit spinor structures locally [5, 6]. The generalized Weyl gauge group [7] thus operates on any spin spaces in a way that does not depend at all upon the action of the relevant spacetime mapping group, and thereby affords the geometric specifications of any spin objects.

Within the $\gamma\varepsilon$-framework, the crucial assumption underlying the settlement of any metric and affine structures amounts to taking at the outset all Hermitian connecting objects, as well as any ε-metric spinors, as covariantly constant entities in both the formalisms. It was shown in Refs. [2, 3] that the implementation

of this assumption actually produces in either formalism a self-consistent set of world-spin metric and affine correlations. The famous Schouten's theory of spin densities [8, 9] turned out, then, to be inherently rooted into the whole structure of the ε-formalism. The situation concerning the obtainment of the curvature spinors for the ε-formalism was slightly touched upon in Refs. [1, 10, 11], but a fairly complete description of spin curvatures was given much later in Refs. [2, 4]. It thus appeared that, in either formalism, spin curvatures should arise formally from a bivector decomposition of mixed world-spin quantities which typically result out of the local action of covariant derivative commutators on Hermitian connecting objects.

The Infeld-van der Waerden formalisms were over the years utilized by many authors not only to reconstruct classical geometric structures, but also to transcribe classification schemes for world curvature tensors and to treat Einstein's equations [12–29]. The viability of all such works relies upon the achievement of systematic developments of the computational methods involved in the original framework. Amongst these developments, in particular, is the attainment of a set of algebraic expansions and valence-reduction devices associated with general spinor-index configurations, which might be directly applied to any curved spacetimes because of their intrinsic symbolic character [12]. Notwithstanding the fact that the formal description of curvature spinors is implicitly carried by the formalisms, the spin curvatures that occur in some of the geometric reconstructions we have just mentioned, were obtained in an artificial way by performing straightforward spinor translations of Riemann and Weyl tensors. The corresponding translational procedures have notably led to the earliest spinor version of Einstein's equations [12, 16]. However, as in the case of the popular spin-coefficient Newman-Penrose treatment of Einstein's equations [12, 20], the gauge structure inherently borne by the formalisms was entirely ruled out and the utmost importance of spin densities apparently neglected.

Indeed, the most striking physical feature of any curvature spinors lies over the fact that they are split as sums of purely gravitational and electromagnetic contributions which bring forth in an inextricably geometric fashion the occurrence of wave functions for gravitons and photons of both handednesses. The gravitational contributions for the ε-formalism had been employed in Refs. [12, 16] to support the transcriptions performed thereabout, but it had already been established in Ref. [29] that any of them should be looked upon as a spinor pair which must be associated to the irreducible decomposition of a Riemann tensor. Gravitational wave functions for both formalisms are accordingly defined as totally symmetric curvature pieces that occur in spinor decompositions of Weyl tensors. Any electromagnetic curvature contribution, on the other hand, amounts to a pair of suitably contracted pieces that enter the spinor representation of a locally defined Maxwell bivector, while satisfying a peculiar conjugation property [2, 4]. The eventually occurrent covariant constancy of the metric spinors for the γ-formalism thus makes up the ultimate condition for the absence of electromagnetic curvatures from the spacetime geometry [1, 10, 15].

The propagation in vacuum of ε-formalism gravitons and their couplings to *external* electromagnetic fields were described in Refs. [12, 16], whereas the $\gamma\varepsilon$-descriptions of the couplings between gravitons and Infeld-van der Waerden photons were exhibited for the first time by the work of Ref. [2] in conjunction with the derivation of the overall system of sourceless $\gamma\varepsilon$-wave equations. It was then found that, in both formalisms, any couplings between gravitons and $\gamma\varepsilon$-photons are strictly borne by the wave equations that govern the electromagnetic propagation. In Ref. [30], it was suggested that a description of some of the physical properties of the cosmic microwave background may be achieved by looking at the propagation in Friedmann-like conformally flat spacetimes of Infeld-van der Waerden photons. As carried out by the work of Ref. [2], the explicit derivation of the wave equations for gravitons was based upon the combination of a generalized spinor version of the gravitational Bianchi identity with certain calculational techniques and Einstein's equations in vacuum. All the statements brought about by the derivation procedures appeared, in effect, to bear invariance under transformations belonging to the Weyl gauge group, and the gravitational Bianchi identity effectively supplied the formal field equations for gravitons. In the ε-formalism, those techniques take up suitably contracted second-order covariant-derivative operators which bring forward the geometric specification of the action of covariant-derivative commutators on arbitrary spin densities.

The introduction of sources into the $\gamma\varepsilon$-descriptions of gravitons was carried out originally in Ref. [31] where it was shown with the help of the same calculational techniques as the ones which had been utilized in the sourceless context, that the second-order covariant derivative of any energy-momentum tensor held up by the full Einstein's equations must enjoy a general total-symmetry property related to the energy-momentum conservation law of general relativity. A work aimed at describing the effect on the propagation of gravitons of sources coming from electromagnetic curvatures, was carried out afterwards in Ref. [32]. In this latter work, all the key electromagnetic patterns that should take place on the right-hand sides of the wave equations for gravitons in both formalisms, were clearly obtained.

The leading purpose of the present paper is to review in a concise way the formulation of the sourceful theory of gravitons as referred to above, while placing a considerable deal of enhancement on the spinor symmetries tied in with the inner structure of general relativity. Our work constitutes the most comprehensive presentation of the traditional two-component spinor formulation of general relativity, and it is essentially upon this perspective that the main motivation for having elaborated our review rests. Nonetheless, we will by this point restrict ourselves to bringing out only the ε-formalism configurations since this attitude may enable us to achieve our goals in a more transparent manner.

For the sake of consistency, some basic world-spin definitions are given in Section 2. In Section 3, we will call upon the curvature structures for the formalism

taken into consideration without recalling any spin-affine prescriptions. There, we will have to define typical electromagnetic wave functions but the theory of Infeld-van der Waerden photons [2, 4] shall not be developed. Section 4 replicates the ε-formalism part of the calculational techniques posed in Ref. [31]. The pertinent field and wave equations are brought together in Section 5. In Section 6, we make some remarks on the general relativity framework and set forth the sourceless limiting case. The inherent spin-density character of the ε-formalism, which we had alluded to previously, will be taken for granted to the extent that we shall specify the gauge behaviours of the ε-formalism entities to be allowed for herein without displaying beforehand Schouten's theory of spin densities.

We should point up that, as far as the development of our work is concerned, any wave functions must be regarded physically as classical entities. The notation adhered to in Ref. [2] will be used throughout the paper except that spacetime components will now be labelled by lower-case Greek letters. The works of Refs. [1] through [4] shall be utilized many times in what follows, but we will refer to them explicitly just a few times.

2 Some Basic Settings

We consider a curved spacetime equipped with a symmetric metric tensor $g_{\mu\nu}$ of signature $(+---)$ together with a torsionless covariant derivative operator ∇_μ that satisfies the metric compatibility condition

$$\nabla_\mu g_{\lambda\sigma} = 0. \tag{1}$$

The Riemann tensor $R_{\mu\nu\lambda\sigma}$ of ∇_μ possesses the general skew symmetries[1]

$$R_{\mu\nu\lambda\sigma} = R_{[\mu\nu][\lambda\sigma]}, \tag{2}$$

and satisfies the Bianchi identity $\nabla_{[\mu}R_{\nu\lambda]\sigma\rho} = 0$. It also obeys the cyclic identity $R_{[\mu\nu\lambda]\sigma} = 0$ such that the index-pair symmetry $R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$ likewise holds.[2] So, the Ricci tensor[3] of ∇_μ

$$R_{\mu\nu} = R_{\mu\tau\nu}{}^{\tau} \tag{3}$$

bears symmetry, whence $R_{\mu\nu\lambda\sigma}$ carries $36 - 16$ degrees of freedom while $R_{\mu\nu}$ carries $16 - 6$.

It is useful to reexpress the cyclic and Bianchi identities as

$${}^*R^{\lambda}{}_{\mu\nu\lambda} = 0, \ \nabla^{\rho *}R_{\rho\mu\lambda\sigma} = 0, \tag{4}$$

[1] The whole information on $R_{\mu\nu\lambda\sigma}$ can be particularly extracted from the world-covariant commutator $[\nabla_\mu, \nabla_\nu]u^\lambda = R_{\mu\nu\sigma}{}^{\lambda}u^\sigma$, with u^λ being a world vector.

[2] In Ref. [2], this was posed in a somewhat unclear way.

[3] Our sign convention for $R_{\mu\nu}$ is the same as that adopted in Ref. [12].

where

$$^{*}R_{\mu\nu\lambda\sigma} = \frac{1}{2}\sqrt{-\mathfrak{g}}\mathfrak{e}_{\mu\nu\rho\tau}R^{\rho\tau}{}_{\lambda\sigma} \tag{5}$$

defines a first-left dual, with $\mathfrak{g}$ denoting the determinant of $g_{\mu\nu}$ and $\mathfrak{e}_{\mu\nu\rho\tau}$ standing for one of the invariant alternating Levi-Civitta world densities. The full Einstein's equations will be taken as[4]

$$\Xi_{\mu\nu} = \frac{1}{2}\kappa(T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu}), \tag{6}$$

where $T_{\mu\nu}$ amounts to the world version of the energy-momentum tensor of some sources, κ stands for the Einstein gravitational constant and $T = g^{\lambda\sigma}T_{\lambda\sigma}$ is the metric trace of $T_{\mu\nu}$. The quantity $(-2)\Xi_{\mu\nu}$ is by definition identified with the trace-free part of $R_{\mu\nu}$, that is to say,

$$(-2)\Xi_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}, \tag{7}$$

with $R = g^{\lambda\sigma}R_{\lambda\sigma}$ being the Ricci scalar of ∇_{μ}. In Eq. (6), the cosmological constant λ is allowed for implicitly through the equations

$$G_{\mu\nu} + \lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{8}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{9}$$

is the Einstein tensor, whence we can make use of the extended trace relation[5]

$$R = 4\lambda + \kappa T. \tag{10}$$

The elements of the Weyl gauge group are non-singular complex (2×2)-matrices whose components are defined as

$$\Lambda_{A}{}^{B} = \sqrt{\rho}\exp(i\theta)\delta_{A}{}^{B}, \tag{11}$$

where $\delta_{A}{}^{B}$ denotes the Kronecker symbol, ρ stands for a positive-definite differentiable real-valued function of some local coordinates and θ amounts to the gauge parameter of the group which is usually taken as an arbitrary differentiable real-valued function in spacetime. For the determinant of $(\Lambda_{A}{}^{B})$, we have the expression

$$\det(\Lambda_{A}{}^{B}) = \Delta_{\Lambda} = \rho\exp(2i\theta). \tag{12}$$

[4]Equation (6) effectively encompasses all the four cases of presence-absence of sources and presence-absence of the cosmological constant. It was given originally as Eq. (4.6.32) of Ref. [12].

[5]The tensor $G_{\mu\nu}$ equals the trace-reversal part of $R_{\mu\nu}$. In the absence of the cosmological constant, it obeys the conservation law $\nabla^{\mu}G_{\mu\nu} = 0$.

By definition, any connecting objects for the ε-formalism satisfy anticommutation relations of the type

$$\Sigma_{\mu AA'}\Sigma_{\nu}^{BA'} + \Sigma_{\nu AA'}\Sigma_{\mu}^{BA'} = \delta_A{}^B g_{\mu\nu}, \tag{13}$$

which yield the Hermitian relationships

$$g_{\mu\nu} = \Sigma_{\mu}^{AA'}\Sigma_{\nu}^{BB'}\varepsilon_{AB}\varepsilon_{A'B'},\ \varepsilon_{AB}\varepsilon_{A'B'} = \Sigma_{AA'}^{\mu}\Sigma_{BB'}^{\nu}g_{\mu\nu}, \tag{14}$$

and thence likewise produce the property[6]

$$\Sigma_{\mu A'}^{(A}\Sigma_{\nu}^{B)A'} = \Sigma_{A'[\mu}^{(A}\Sigma_{\nu]}^{B)A'} = \Sigma_{A'[\mu}^{A}\Sigma_{\nu]}^{BA'}. \tag{15}$$

All Hermitian Σ-objects enter the metric picture as invariant spin-tensor densities of absolute weights ± 1. For instance,

$$\Sigma_{\mu}'^{AA'} = \rho\Sigma_{\mu}^{BB'}\Lambda_B^{-1A}\Lambda_{B'}^{-1A'} = \Sigma_{\mu}^{AA'}. \tag{16}$$

The spinors $(\varepsilon_{AB}, \varepsilon_{A'B'})$ as well as their inverses $(\varepsilon^{AB}, \varepsilon^{A'B'})$ are skew objects and constitute the only metric spinors for the ε-formalism. This uniqueness property just stems from the transformation laws

$$\begin{aligned} \varepsilon'^{AB} &= \Delta_\Lambda \varepsilon^{CD}\Lambda_C^{-1A}\Lambda_D^{-1B} = \varepsilon^{AB}, \\ \varepsilon'_{AB} &= (\Delta_\Lambda)^{-1}\Lambda_A{}^C\Lambda_B{}^D\varepsilon_{CD} = \varepsilon_{AB}, \end{aligned} \tag{17}$$

and their complex conjugates, which tell us that the ε-metric spinors are invariant spin-tensor densities of weights and antiweights ± 1. Thus, the spinor structure that represents $\mathfrak{e}_{\mu\nu\lambda\sigma}$ is spelt out as [12, 33]

$$\sqrt{-\mathfrak{g}}\mathfrak{e}_{AA'BB'CC'DD'} = i(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - \text{c.c.}) \tag{18}$$

or, equivalently, as[7]

$$\sqrt{-\mathfrak{g}}\mathfrak{e}_{\mu\nu\lambda\sigma} = i(\Sigma_{\mu AB'}\Sigma_{\nu BA'}\Sigma_{\lambda}^{AA'}\Sigma_{\sigma}^{BB'} - \text{c.c.}), \tag{19}$$

with the symbol "c.c." denoting here as elsewhere an overall complex conjugate piece. The ε-spinors can be seen as Levi-Civitta densities that obey the relations

$$\varepsilon_{[AB}\varepsilon_{C]D} \equiv 0,\ \varepsilon_{A(B}\varepsilon_{C)D} = \varepsilon_{B(A}\varepsilon_{D)C}, \tag{20}$$

whence we also have

$$\Delta_\Lambda = \frac{1}{2}\varepsilon^{AB}\Lambda_A{}^C\Lambda_B{}^D\varepsilon_{CD}. \tag{21}$$

[6] We have sometimes displaced the world index of a connecting object just for an occasional convenience.

[7] Any connecting object bears a world-vector character such that the metric spinors are world invariant quantities as reflected by Eqs. (14), (18) and (19).

3 Curvature Spinors

The information on the overall ε-curvature splitting associated to ∇_μ may emerge from

$$[\nabla_\mu, \nabla_\nu]\Sigma^{\lambda DD'} \doteqdot 2\nabla_{[\mu}(\nabla_{\nu]}\Sigma^{\lambda DD'}) = 0. \tag{22}$$

Upon expanding the middle configuration of (22), we get

$$(\Sigma^{\lambda AB'} W_{\mu\nu A}{}^{B} + \text{c.c.}) + \Sigma^{\sigma BB'} R_{\mu\nu\sigma}{}^{\lambda} = 0, \tag{23}$$

where the quantity $W_{\mu\nu AB}$ defines an ε-formalism mixed world-spin object. It follows that, if (23) is transvected with $\Sigma_{\lambda DB'}$, we will obtain

$$2W_{\mu\nu A}{}^{B} + \delta_A{}^{B} W_{\mu\nu A'}{}^{A'} = \Sigma^{\lambda}_{AB'}\Sigma^{\sigma BB'} R_{\mu\nu\lambda\sigma}, \tag{24}$$

whence it can be stated that

$$4\text{Re}W_{\mu\nu C}{}^{C} = R_{\mu\nu\lambda}{}^{\lambda} \equiv 0, \tag{25}$$

which means that the trace $W_{\mu\nu C}{}^{C}$ amounts to a purely imaginary quantity.

In Refs. [1, 2, 4], it becomes manifest that

$$W_{\mu\nu[CD]} = \frac{1}{2}\varepsilon_{CD} W_{\mu\nu M}{}^{M} = -iF_{\mu\nu}\varepsilon_{CD}, \tag{26}$$

with $F_{\mu\nu}$ being a Maxwell bivector, whereas the piece

$$W_{\mu\nu(CD)} = \frac{1}{2}\Sigma^{\lambda}_{CB'}\Sigma^{\sigma B'}_{D} R_{\mu\nu\lambda\sigma} \tag{27}$$

should supply the gravitational contribution carried by $W_{\mu\nu CD}$. It must be stressfully emphasized that Eq. (15) really ensures the genuineness of the spinor symmetry exhibited by (27). One then obtains the splitting

$$W_{\mu\nu CD} = \frac{1}{2}\Sigma^{\lambda}_{CB'}\Sigma^{\sigma B'}_{D} R_{\mu\nu\lambda\sigma} - iF_{\mu\nu}\varepsilon_{CD}, \tag{28}$$

which thus behaves as an invariant spin-tensor density of weight -1, namely,

$$W'_{\mu\nu CD} = (\Delta_\Lambda)^{-1}\Lambda_C{}^{L}\Lambda_D{}^{M} W_{\mu\nu LM} = W_{\mu\nu CD}. \tag{29}$$

Evidently, the index contraction that yields (25) from (24) brings about annihilation of the gravitational information carried by $W_{\mu\nu CD}$ whilst a symmetrization over the indices C and D of (28) similarly causes annihilation of the electromagnetic information. The primary ε-curvature spinors of ∇_μ arise from the decomposition of the bivector configuration of (28). We have, in effect,

$$\Sigma^{\mu}_{AA'}\Sigma^{\nu}_{BB'} W_{\mu\nu CD} = \varepsilon_{A'B'}\omega_{ABCD} + \varepsilon_{AB}\omega_{A'B'CD}, \tag{30}$$

where

$$\omega_{ABCD} = \omega_{(AB)CD} \doteq \frac{1}{2}\Sigma^{\mu}_{AA'}\Sigma^{\nu A'}_{B} W_{\mu\nu CD} \tag{31}$$

and

$$\omega_{A'B'CD} = \omega_{(A'B')CD} \doteq \frac{1}{2}\Sigma^{\mu}_{AA'}\Sigma^{\nu A}_{B'} W_{\mu\nu CD}. \tag{32}$$

Due to the gauge character specified by (29), the ω-spinors are subject to the laws

$$\omega'_{ABCD} = (\Delta_\Lambda)^{-2}\Lambda_A{}^L\Lambda_B{}^M\Lambda_C{}^R\Lambda_D{}^S\omega_{LMRS} = \omega_{ABCD} \tag{33}$$

and

$$\omega'_{A'B'CD} = \rho^{-2}\Lambda_{A'}{}^{L'}\Lambda_{B'}{}^{M'}\Lambda_C{}^R\Lambda_D{}^S\omega_{L'M'RS} = \omega_{A'B'CD}. \tag{34}$$

The world gravitational curvatures of ∇_μ can be completely reinstated from the pair

$$\mathbf{G} = (\omega_{AB(CD)},\ \omega_{A'B'(CD)}). \tag{35}$$

Thus, the elements of the $\mathbf{G}$-pair occur in the spinor expression for $R_{\mu\nu\lambda\sigma}$ according to the prescription [12, 29]

$$R_{AA'BB'CC'DD'} = (\varepsilon_{A'B'}\varepsilon_{C'D'}\omega_{AB(CD)} + \varepsilon_{AB}\varepsilon_{C'D'}\omega_{A'B'(CD)}) + \text{c.c.}. \tag{36}$$

This property was established in Refs. [2, 4] by rewriting Eq. (27) as

$$\frac{1}{2}\Sigma^{\lambda}_{CA'}\Sigma^{\sigma A'}_{D} R_{\mu\nu\lambda\sigma} = \Sigma^{E}_{M'[\mu}\Sigma^{FM'}_{\nu]}\omega_{EF(CD)} + \Sigma^{E'}_{M[\mu}\Sigma^{F'M}_{\nu]}\omega_{E'F'(CD)}, \tag{37}$$

likewise utilizing the metric formulae

$$\Sigma^{\mu}_{AM'}\Sigma^{\nu M'}_{B}\Sigma^{EA'}_{[\mu}\Sigma^{F}_{\nu]A'} = -2\varepsilon^{(E}{}_A\varepsilon^{F)}{}_B \tag{38}$$

and

$$\Sigma^{\mu}_{AM'}\Sigma^{\nu M'}_{B}\Sigma^{ME'}_{[\mu}\Sigma^{F'}_{\nu]M} = \varepsilon_{(AB)}\varepsilon^{E'F'} \equiv 0, \tag{39}$$

together with the complex conjugates of (38) and (39). We may then pick up the individual ω-spinors of (36) in agreement with the coupling schemes

$$\begin{aligned}\frac{1}{2}\Sigma^{\mu}_{AM'}\Sigma^{\nu M'}_{B}\Sigma^{\lambda}_{CA'}\Sigma^{\sigma A'}_{D} R_{\mu\nu\lambda\sigma} &= -\Sigma^{\mu}_{AM'}\Sigma^{\nu M'}_{B}\Sigma^{EA'}_{[\mu}\Sigma^{F}_{\nu]A'}\omega_{EF(CD)} \\ &= 2\omega_{AB(CD)}\end{aligned} \tag{40}$$

and

$$\begin{aligned}\frac{1}{2}\Sigma^{\mu}_{MA'}\Sigma^{\nu M}_{B'}\Sigma^{\lambda}_{CM'}\Sigma^{\sigma M'}_{D} R_{\mu\nu\lambda\sigma} &= -\Sigma^{\mu}_{MA'}\Sigma^{\nu M}_{B'}\Sigma^{AE'}_{[\mu}\Sigma^{F'}_{\nu]A}\omega_{E'F'(CD)} \\ &= 2\omega_{A'B'(CD)}.\end{aligned} \tag{41}$$

The spinor symmetries carried by the entries of the **G**-pair correspond to the skew symmetry in the indices of the pairs $\mu\nu$ and $\lambda\sigma$ borne by $R_{\mu\nu\lambda\sigma}$. In view of the index-pair symmetry of $R_{\mu\nu\lambda\sigma}$, one should demand the additional symmetries

$$\omega_{AB(CD)} = \omega_{(CD)AB},\ \omega_{A'B'(CD)} = \omega_{(CD)A'B'}, \tag{42}$$

whence the only symmetries of any gravitational spinors look like

$$\begin{aligned} \omega_{AB(CD)} &= \omega_{(AB)(CD)} = \omega_{(CD)(AB)}, \\ \omega_{A'B'(CD)} &= \omega_{(A'B')(CD)} = \omega_{(CD)(A'B')}. \end{aligned} \tag{43}$$

Therefore, the second entry of the **G**-pair has to be regarded as a symmetric trace-free Hermitian entity and, consequently, there is no fixed prescription for ordering the indices of $\omega_{A'B'(CD)}$. We can thus write the tracelessness relation

$$g_{\mu\nu}\Sigma^{\mu CA'}\Sigma^{\nu DB'}\omega_{A'B'(CD)} = 0. \tag{44}$$

A cosmological interpretation of the gravitational spinors can be accomplished by working out Eq. (43) and invoking Einstein's equations. In this connection, it is convenient to reexpress (36) as

$$R_{AA'BB'CC'DD'} = (\varepsilon_{A'B'}\varepsilon_{C'D'}\mathrm{X}_{ABCD} + \varepsilon_{AB}\varepsilon_{C'D'}\Xi_{CA'DB'}) + \text{c.c.}, \tag{45}$$

with the XΞ-spinors being defined by

$$\mathrm{X}_{ABCD} \doteqdot \frac{1}{4}\varepsilon^{A'B'}\varepsilon^{C'D'}R_{AA'BB'CC'DD'} = \omega_{AB(CD)} \tag{46}$$

and

$$\Xi_{CA'DB'} \doteqdot \frac{1}{4}\varepsilon^{AB}\varepsilon^{C'D'}R_{AA'BB'CC'DD'} = \omega_{A'B'(CD)}. \tag{47}$$

The symmetries of the X-spinor promptly yield the statement

$$\varepsilon^{AD}\mathrm{X}_{A(BC)D} = 0 \Leftrightarrow \varepsilon^{BC}\mathrm{X}_{B(AD)C} = 0, \tag{48}$$

which, in turn, produces the relations

$$\varepsilon^{AD}\mathrm{X}_{ABCD} = \chi\varepsilon_{BC},\ \varepsilon^{BC}\mathrm{X}_{ABCD} = \chi\varepsilon_{AD}, \tag{49}$$

with

$$\mathrm{X}_{AB}{}^{AB} = 2\chi, \tag{50}$$

and χ standing for a world-spin invariant. By calling for Eqs. (5) and (18), we can write down the first-left dual pattern [12]

$${}^{*}R_{AA'BB'CC'DD'} = [(-i)(\varepsilon_{A'B'}\varepsilon_{C'D'}\mathrm{X}_{ABCD} - \varepsilon_{AB}\varepsilon_{C'D'}\Xi_{CA'DB'})] + \text{c.c.}, \tag{51}$$

which, by virtue of the first of Eqs. (4), leads us to

$$\varepsilon_{A'D'}\varepsilon^{BC}\mathrm{X}_{ABCD} = \varepsilon_{AD}\varepsilon^{B'C'}\mathrm{X}_{A'B'C'D'}. \tag{52}$$

Hence, by coupling both sides of (52) with $\varepsilon^{AD}\varepsilon^{A'D'}$, one arrives at the equality

$$\mathrm{Im}\chi = 0, \tag{53}$$

while Eqs. (45) and (53) immediately give $R = 8\chi$, whence the X-spinor possesses

$$18 - 8 + 1 = 18 - 6 - 1 = 11$$

real independent components. In addition, from (45), we get the spinor expression for $R_{\mu\nu}$

$$R_{AA'BB'} = \frac{1}{4}R\varepsilon_{AB}\varepsilon_{A'B'} - 2\Xi_{AA'BB'}, \tag{54}$$

such that, by recalling Eqs. (6) and (7), we see that the Ξ-spinor satisfies the full Einstein's equations

$$\Xi_{AA'BB'} = \frac{1}{2}\kappa(T_{AA'BB'} - \frac{1}{4}T\varepsilon_{AB}\varepsilon_{A'B'}), \tag{55}$$

while

$$\Xi_{\mu\nu} = \Sigma_{\mu}^{AA'}\Sigma_{\nu}^{BB'}\Xi_{AA'BB'} \tag{56}$$

and $\Xi_{\mu}{}^{\mu} = 0$, in compatibility with (44) and (47). Then, the Ξ-spinor possesses

$$18 - 8 - 1 = 4 + 6 - 1 = 10 - 1 = 9$$

real independent components such that the numbers of degrees of freedom of the Riemann and Ricci tensors are recovered[8] as $11 + 9$ and $1 + 9$, respectively. We mention in passing that, when only traceless sources are present, the spinor expression for the Einstein tensor appears as

$$G_{AA'BB'} = -2\Xi_{AA'BB'} - \lambda\varepsilon_{AB}\varepsilon_{A'B'}. \tag{57}$$

The symmetries of X_{ABCD} considerably simplify the four-index reduction formula [12]

$$\begin{aligned}\mathrm{X}_{ABCD} = &\mathrm{X}_{(ABCD)} - \frac{1}{4}(\varepsilon_{AB}\mathrm{X}^{M}{}_{(MCD)} + \varepsilon_{AC}\mathrm{X}^{M}{}_{(MBD)} + \varepsilon_{AD}\mathrm{X}^{M}{}_{(MBC)}) \\ &- \frac{1}{3}(\varepsilon_{BC}\mathrm{X}^{M}{}_{A(MD)} + \varepsilon_{BD}\mathrm{X}^{M}{}_{A(MC)}) - \frac{1}{2}\varepsilon_{CD}\mathrm{X}_{AB}{}^{M}{}_{M}.\end{aligned} \tag{58}$$

When combined with (48) and (49), this property provides us with the expansion

$$\mathrm{X}_{ABCD} = \mathrm{X}_{(ABCD)} - \frac{1}{12}R\varepsilon_{A(C}\varepsilon_{D)B}, \tag{59}$$

[8] The prescription $11 + 9$ was originally given in Ref. [29].

which recovers the number of real independent components of X_{ABCD} as $10+1$, with

$$\mathrm{X}_{(ABCD)} = \mathrm{X}_{A(BCD)} = \mathrm{X}_{(ABC)D}. \tag{60}$$

In case $T = 0$, we may rewrite (59) as

$$\mathrm{X}_{ABCD} = \mathrm{X}_{(ABCD)} - \frac{1}{3}\lambda\varepsilon_{A(C}\varepsilon_{D)B}. \tag{61}$$

It is of some interest to notice that the Hermitian configuration

$$\begin{aligned}&(\varepsilon_{A(C}\varepsilon_{D)B}\varepsilon_{A'B'}\varepsilon_{C'D'}) + \text{c.c.}\\ &= \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'} - \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'}\end{aligned} \tag{62}$$

gives rise to

$$\begin{aligned}&\varepsilon_{A'B'}\varepsilon_{C'D'}(\mathrm{X}_{(ABCD)} - \mathrm{X}_{ABCD}) + \text{c.c.}\\ &= \frac{1}{3}\lambda(\varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'} - \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'}).\end{aligned} \tag{63}$$

The electromagnetic spinors carried by $W_{\mu\nu CD}$ comprises the constituents of the bivector configuration of (26). Formally, we have the contracted pair

$$\mathbf{E} = (\omega_{ABC}{}^{C},\ \omega_{A'B'C}{}^{C}), \tag{64}$$

whose entries thus enter the bivector decomposition

$$\Sigma^{\mu}_{AA'}\Sigma^{\nu}_{BB'}F_{\mu\nu} = \frac{1}{2}i(\varepsilon_{A'B'}\omega_{ABC}{}^{C} + \varepsilon_{AB}\omega_{A'B'C}{}^{C}), \tag{65}$$

and likewise supply the wave functions

$$\phi_{AB} = \frac{1}{2}i\omega_{ABC}{}^{C},\ \phi_{A'B'} = \frac{1}{2}i\omega_{A'B'C}{}^{C}. \tag{66}$$

We are then led to the general curvature splittings

$$\omega_{ABCD} = \omega_{(AB)(CD)} + \frac{1}{2}\omega_{(AB)M}{}^{M}\varepsilon_{CD} \tag{67}$$

and

$$\omega_{A'B'CD} = \omega_{(A'B')(CD)} + \frac{1}{2}\omega_{(A'B')M}{}^{M}\varepsilon_{CD}, \tag{68}$$

along with their complex conjugates. It is shown in Refs. [2,4] that the spinors of (64) obey the peculiar conjugacy relations

$$\omega_{ABC}{}^{C} = -\omega_{ABC'}{}^{C'},\ \omega_{A'B'C}{}^{C} = -\omega_{A'B'C'}{}^{C'}. \tag{69}$$

From Eqs. (33) and (34), we see that the electromagnetic curvatures are invariant spin-tensor densities subject to the laws

$$\omega'_{ABC}{}^{C} = (\Delta_{\Lambda})^{-1}\Lambda_{A}{}^{L}\Lambda_{B}{}^{M}\omega_{LMC}{}^{C} = \omega_{ABC}{}^{C} \tag{70}$$

and

$$\omega'_{A'B'C}{}^{C} = (\bar{\Delta}_{\Lambda})^{-1}\Lambda_{A'}{}^{L'}\Lambda_{B'}{}^{M'}\omega_{L'M'C}{}^{C} = \omega_{A'B'C}{}^{C}. \tag{71}$$

4 Calculational Techniques

We shall now present the calculational devices which will indeed pave the way for the development of Section 5. The starting device is written out as the commutator

$$\Sigma^{\mu}_{AA'}\Sigma^{\nu}_{BB'}[\nabla_{\mu},\nabla_{\nu}]=\varepsilon_{A'B'}\Delta_{AB}+\varepsilon_{AB}\Delta_{A'B'}, \tag{72}$$

with each of the Δ-kernels amounting to a symmetric second-order differential operator that bears linearity as well as the Leibniz-rule property. More explicitly, we have the defining patterns

$$\Delta_{AB}=\nabla_{C'(A}\nabla^{C'}_{B)},\ \Delta_{A'B'}=\nabla_{C(A'}\nabla^{C}_{B')}, \tag{73}$$

which behave as gauge-invariant spin-tensor densities of weight -1 and anti-weight -1, respectively. The contravariant form of the Δ-operators is supplied by

$$\Delta^{AB}=\varepsilon^{AC}\varepsilon^{BD}\Delta_{CD}, \tag{74}$$

which immediately produces the configuration[9]

$$\Delta^{AB}=\nabla^{(A}_{C'}\nabla^{B)C'}=-\nabla^{C'(A}\nabla^{B)}_{C'}. \tag{75}$$

When acting on a world-spin scalar, any Δ-operator gives an identically vanishing result.[10]

The object $W_{\mu\nu CD}$ can be alternatively obtained from either of the commutators

$$[\nabla_{\mu},\nabla_{\nu}]\zeta^{A}=W_{\mu\nu M}{}^{A}\zeta^{M},\ [\nabla_{\mu},\nabla_{\nu}]\eta_{A}=-W_{\mu\nu A}{}^{M}\eta_{M}, \tag{76}$$

with ζ^{A} and η_{A} being some spin vectors. It turns out that one can write the following derivatives

$$\Delta_{AB}\zeta^{C}=\omega_{ABD}{}^{C}\zeta^{D},\ \Delta_{A'B'}\zeta^{C}=\omega_{A'B'D}{}^{C}\zeta^{D} \tag{77}$$

and

$$\Delta_{AB}\eta_{C}=-\omega_{ABC}{}^{D}\eta_{D},\ \Delta_{A'B'}\eta_{C}=-\omega_{A'B'C}{}^{D}\eta_{D}. \tag{78}$$

Invoking (67) and (68) then yields the formulae

$$\Delta_{AB}\zeta^{C}=\mathrm{X}_{ABD}{}^{C}\zeta^{D}+\frac{1}{2}\omega_{ABD}{}^{D}\zeta^{C} \tag{79}$$

and

$$\Delta_{A'B'}\zeta^{C}=\Xi_{A'B'D}{}^{C}\zeta^{D}+\frac{1}{2}\omega_{A'B'D}{}^{D}\zeta^{C}, \tag{80}$$

[9] The derivation of (75) includes taking up the covariant constancy of the ε-spinors.

[10] In essence, this property reflects the torsionlessness of ∇_{μ}.

together with their complex conjugates and lower-index versions. For a complex spin-scalar density α of weight w, one has the derivatives

$$\Delta_{AB}\alpha = -w\alpha\omega_{ABC}{}^{C},\ \Delta_{A'B'}\alpha = -w\alpha\omega_{A'B'C}{}^{C}, \tag{81}$$

which come right away from the integrability condition [4]

$$[\nabla_\mu, \nabla_\nu]\alpha = 2iw\alpha F_{\mu\nu}. \tag{82}$$

Hence, the prototype for Δ-derivatives of covariant spin-tensor densities, say, is prescribed as the expansion

$$\Delta_{AB}(\alpha U_{C\ldots D}) = \alpha(\Delta_{AB}U_{C\ldots D} - w\omega_{ABM}{}^{M}U_{C\ldots D}), \tag{83}$$

with $U_{C\ldots D}$ being a spin tensor.

There are some situations wherein the calculation of Δ-derivatives can be performed as if electromagnetic pieces were absent from curvature splittings such that the differential expansions would then involve only purely gravitational contributions. The first consideration concerning this observation is that there would occur a cancellation of those pieces whenever Δ-derivatives of Hermitian quantities are actually computed. Such a cancellation would also happen when we let Δ-operators act freely upon spin tensors having the same numbers of contravariant and covariant indices. For $w < 0$, it would still occur in the expansion (83) when $U_{C\ldots D}$ carries $-2w$ indices and $\mathrm{Im}\alpha \neq 0$ everywhere. A similar property likewise holds for cases that involve outer products between contravariant spin tensors and complex spin-scalar densities having suitable positive weights. We shall see in the forthcoming Section that such properties neatly fit in with the case of gravitational wave functions. Also, the derivation of the wave equations to be shown in the next Section, takes particular account of the differential splittings

$$\nabla^{C}_{A'}\nabla^{AA'} = \Delta^{AC} + \frac{1}{2}\varepsilon^{CA}\Box,\ \nabla^{A'}_{C}\nabla_{AA'} = \frac{1}{2}\varepsilon_{AC}\Box - \Delta_{AC}, \tag{84}$$

along with the simpler algebraic rules

$$\begin{aligned} 2\nabla^{A'}_{[C}\nabla_{A]A'} &= \nabla^{A'}_{D}(\varepsilon_{CA}\nabla^{D}_{A'}) = \varepsilon_{AC}\Box, \\ 2\nabla^{[C}_{A'}\nabla^{A]A'} &= \nabla^{D}_{A'}(\varepsilon^{AC}\nabla^{A'}_{D}) = \varepsilon^{CA}\Box \end{aligned} \tag{85}$$

and the world-spin invariant definition

$$\Box \doteqdot \Sigma^{\mu}_{DD'}\Sigma^{\nu DD'}\nabla_\mu\nabla_\nu = \nabla_\mu\nabla^\mu. \tag{86}$$

5 Field and Wave Equations for Gravitons

The totally symmetric piece that occurs in Eq. (59) defines a prototype wave function for gravitons, namely,

$$\Psi_{ABCD} \doteqdot \mathrm{X}_{(ABCD)}. \tag{87}$$

At each non-singular spacetime point, Ψ_{ABCD} represents the ten independent degrees of freedom of $g_{\mu\nu}$, and comes into play as a massless uncharged field of spin ± 2 which is carried by the curvature structure of a non-conformally flat spacetime. When taken together with its complex conjugate, the Ψ-wave function (87) thus enters the spinor expression for the Weyl tensor $C_{\mu\nu\lambda\sigma}$ of ∇_μ in accordance with the expansion [12, 16]

$$\Sigma^{\mu}_{AA'}\Sigma^{\nu}_{BB'}\Sigma^{\lambda}_{CC'}\Sigma^{\sigma}_{DD'}C_{\mu\nu\lambda\sigma} = \varepsilon_{A'B'}\varepsilon_{C'D'}\Psi_{ABCD} + \text{c.c.}. \tag{88}$$

To derive the relevant field equations, one has to utilize the expression (51) for working out the coupled Bianchi identities

$$\varepsilon^{C'D'}\nabla^{AA'*}R_{AA'BB'CC'DD'} = 0 \tag{89}$$

and

$$\varepsilon^{CD}\nabla^{AA'*}R_{AA'BB'CC'DD'} = 0. \tag{90}$$

We have, in effect,

$$\nabla^{A}_{B'}\mathrm{X}_{ABCD} = \nabla^{A'}_{B}\Xi_{CDA'B'} \tag{91}$$

whence, performing a symmetrization over the indices B, C and D of (91) and recalling the property (60), yields the statement

$$\nabla^{A}_{B'}\Psi_{ABCD} = \nabla^{A'}_{(B}\Xi_{CD)A'B'}. \tag{92}$$

Hence, on account of Einstein's equations, we obtain

$$\nabla^{A}_{B'}\Psi_{ABCD} = \frac{1}{2}\kappa\nabla^{A'}_{(B}T_{CD)A'B'}. \tag{93}$$

We notice that the symmetrization occurring in (92) annihilates the T-trace piece of Einstein's equations (see Eq. (55) above). Equation (93) comes about as a sourceful field equation for gravitons. In a similar way, for Ψ^{ABCD}, we get[11]

$$\nabla_{AB'}\Psi^{ABCD} = -\frac{1}{2}\kappa\nabla^{A'(B}T^{CD)}{}_{A'B'}. \tag{94}$$

The gauge behaviour specified by Eq. (33) allows us to say that the valence pattern of $\Psi_{AB}{}^{CD}$ yields an invariant *spin-tensor* character. Consequently, if we account for the symmetry of $T_{\mu\nu}$ along with the expansion[12]

$$T_{CDA'B'} = T_{(CD)(A'B')} + \frac{1}{4}T\varepsilon_{CD}\varepsilon_{A'B'}, \tag{95}$$

[11] Equation (33) shows us that Ψ_{ABCD} and Ψ^{ABCD} are invariant spin-tensor densities of weights -2 and $+2$, respectively.

[12] The spinor version of a traceless $T_{\mu\nu}$ enjoys the same absence of index ordering prescription as that of $\omega_{A'B'(CD)}$.

and make use of the properties

$$T_{(CD)A'B'} = T_{(CD)(A'B')},\ T_{CD(A'B')} = T_{(CD)(A'B')}, \tag{96}$$

we will obtain the field equation

$$\nabla^{A}_{B'}\Psi_{AB}{}^{CD} = \frac{1}{6}\kappa(\nabla^{A'}_{B}T^{(CD)}{}_{(A'B')} + 2\nabla^{A'(C}T_{B}{}^{D)}{}_{(A'B')}). \tag{97}$$

In fact, towards carrying through the derivation of the wave equation for any Ψ-field, one must necessarily implement the techniques considered in Section 4. We shall work out only the structures for the field of Eq. (97). The statements for the fields carried by Eqs. (93) and (94) may be attained thereafter either by making use of the covariant constancy of the ε-metric spinors or by utilizing the weight-valence properties of Δ-derivatives mentioned in Section 4. In this Section, we will use the particular convention whereby vertical bars surrounding an index block should mean that the indices singled out are not to partake of a symmetry operation.

Let us rewrite the kernel of the right-hand side of Eq. (97) as

$$\varepsilon^{CM}\varepsilon^{DN}\varepsilon^{B'D'}\nabla^{A'}_{(B}T_{MN)A'D'} = -\varepsilon^{CM}\varepsilon^{DN}\varepsilon^{D'B'}\varepsilon_{S(B}\nabla^{SA'}T_{MN)A'D'}. \tag{98}$$

By operating on (98) with $\nabla^{L}_{B'}$ and using the index-displacement rule

$$\nabla^{L(A'}\nabla^{D')S} = \nabla^{S(A'}\nabla^{D')L} + \varepsilon^{LS}\Delta^{A'D'}, \tag{99}$$

one gets the development

$$\begin{aligned}
&\varepsilon^{CM}\varepsilon^{DN}\varepsilon_{S(B}\nabla^{LD'}\nabla^{SA'}T_{MN)A'D'} \\
&= \varepsilon^{CM}\varepsilon^{DN}\varepsilon_{S(B}\nabla^{L(A'}\nabla^{D')S}T_{MN)A'D'} \\
&= \varepsilon^{CM}\varepsilon^{DN}\varepsilon_{S(B}(\nabla^{S(A'}\nabla^{D')L} + \varepsilon^{LS}\Delta^{A'D'})T_{MN)A'D'} \\
&= \varepsilon^{CM}\varepsilon^{DN}(\nabla^{(A'}_{(B}\nabla^{D')L} + \varepsilon^{L}{}_{(B}\Delta^{A'D'})T_{MN)A'D'}.
\end{aligned} \tag{100}$$

Owing to Eqs. (48) and (96), each of the X-terms taken up by the evaluation of the Δ-derivative of (100) is equal to zero, whereas the involved contracted ω-pieces cancel out one another because of the Hermiticity of $T_{AA'BB'}$. We then end up with

$$\Delta^{A'D'}T_{MNA'D'} = -2\Xi^{AA'D'}{}_{(M}T_{N)AA'D'} = -\kappa T^{AA'D'}{}_{(M}T_{N)AA'D'} \equiv 0, \tag{101}$$

along with the contribution

$$\frac{1}{2}\kappa\varepsilon^{CM}\varepsilon^{DN}\varepsilon_{S(B}\nabla^{LD'}\nabla^{SA'}T_{MN)A'D'} = \frac{1}{2}\kappa\varepsilon^{CM}\varepsilon^{DN}\nabla^{(A'}_{(B}\nabla^{D')L}T_{MN)A'D'}. \tag{102}$$

For the left-hand side of Eq. (97), we utilize the splittings (84) to obtain

$$\nabla^{L}_{B'}\nabla^{AB'}\Psi_{AB}{}^{CD} = (\Delta^{AL} + \frac{1}{2}\varepsilon^{LA}\Box)\Psi_{AB}{}^{CD}. \tag{103}$$

The Δ-derivative of (103) shows up as [4, 31]

$$\begin{aligned}\Delta^{AL}\Psi_{AB}{}^{CD} &= (\mathrm{X}^{AL}{}_{M}{}^{C}\Psi_{AB}{}^{MD} + \mathrm{X}^{AL}{}_{M}{}^{D}\Psi_{AB}{}^{CM}) \\ &\quad - (\mathrm{X}^{AL}{}_{A}{}^{M}\Psi_{MB}{}^{CD} + \mathrm{X}^{AL}{}_{B}{}^{M}\Psi_{AM}{}^{CD}) \\ &= \frac{1}{4}R\Psi^{CDL}{}_{B} - 3Q^{(CDL)M}\varepsilon_{MB} \\ &= \frac{1}{4}R\Psi^{CDL}{}_{B} - 3Q^{(CDLM)}\varepsilon_{MB},\end{aligned} \tag{104}$$

where

$$Q^{CDLM} \doteqdot \Psi_{GH}{}^{CD}\Psi^{LMGH}. \tag{105}$$

Therefore, by combining together Eqs. (102)-(105), after lowering the index L and making some index rearrangements, we get the equation[13]

$$(\Box + \frac{1}{2}R)\Psi_{AB}{}^{CD} - 6\Psi_{MN}{}^{(CD}\Psi^{GH)MN}\varepsilon_{GA}\varepsilon_{HB} = -\kappa S_{AB}{}^{CD}, \tag{106}$$

with the definition

$$S_{ABCD} = \nabla^{(A'}_{(B}\nabla^{B')}_{|A|}T_{CD)A'B'}. \tag{107}$$

It is evident that the configuration (107) bears symmetry in the indices B, C and D. Its symmetry in A and B, as presumably imposed by Eq. (106), can be easily verified by recalling (96) and (101) to carry out the computation

$$\begin{aligned}&\varepsilon_{L[A}(\nabla^{(A'}_{B]}\nabla^{B')L}T_{CDA'B'} + \nabla^{(A'}_{|C|}\nabla^{B')L}T_{B]DA'B'} + \nabla^{(A'}_{|D|}\nabla^{B')L}T_{B]CA'B'}) \\ &= \frac{1}{2}\varepsilon_{BA}(\Delta^{A'B'}T_{CDA'B'} + \nabla^{(A'}_{C}\nabla^{B')L}T_{LDA'B'} + \nabla^{(A'}_{D}\nabla^{B')L}T_{LCA'B'}) \\ &= \frac{1}{2}\varepsilon_{BA}(\nabla^{A'}_{C}\nabla^{LB'}T_{LB'DA'} + \nabla^{A'}_{D}\nabla^{LB'}T_{LB'CA'}) = \varepsilon_{BA}\nabla^{A'}_{(C}\nabla^{\mu}T_{D)A'\mu} = 0,\end{aligned}$$

with the last step of which being due to the presupposed divergencelessness of $T_{\mu\nu}$. Moreover, this latter symmetry property remains also valid for the indices A, C and A, D. We have, for instance,

$$\begin{aligned}&\varepsilon_{L[A}(\nabla^{(A'}_{|B|}\nabla^{B')L}T_{C]DA'B'} + \nabla^{(A'}_{C]}\nabla^{B')L}T_{BDA'B'} + \nabla^{(A'}_{|D}\nabla^{B')L}T_{B|C]A'B'}) \\ &= \frac{1}{2}\varepsilon_{CA}(\nabla^{(A'}_{B}\nabla^{B')L}T_{LDA'B'} + \Delta^{A'B'}T_{BDA'B'} + \nabla^{(A'}_{D}\nabla^{B')L}T_{BLA'B'}) \\ &= \frac{1}{2}\varepsilon_{CA}(\nabla^{A'}_{B}\nabla^{LB'}T_{LB'DA'} + \nabla^{A'}_{D}\nabla^{LB'}T_{LB'BA'}) = \varepsilon_{CA}\nabla^{A'}_{(B}\nabla^{\mu}T_{D)A'\mu} = 0,\end{aligned}$$

[13] A mistyped sign occurs on the right-hand side of Eq. (44) of Ref. [31] but it was not passed on to the manipulations that led to any key equations. The factor $-\frac{1}{2}\kappa$ borne therein by the right-hand sides of Eqs. (48), (50) and (51) should be replaced with $-\kappa$.

whence S_{ABCD} bears the total-symmetry property

$$S_{ABCD} = \nabla^{(A'}_{(A}\nabla^{B')}_{B}T_{CD)A'B'} = S_{(ABCD)}. \tag{108}$$

As was said before, Ψ_{ABCD} and Ψ^{ABCD} are invariant spin-tensor densities of weights -2 and $+2$, whence they match some of the situations explained in Section 4 regarding eventual absences of electromagnetic contributions from Δ-derivatives. It follows that the left-hand sides of Eqs. (93) and (94) could be treated in a way quite similar to the case of (97). We thus obtain

$$(\Box + \frac{1}{2}R)\Psi_{ABCD} - 6\Psi_{MN(AB}\Psi_{CD)}{}^{MN} = -\kappa\nabla^{(A'}_{(A}\nabla^{B')}_{B}T_{CD)A'B'} \tag{109}$$

and

$$(\Box + \frac{1}{2}R)\Psi^{ABCD} - 6\Psi_{MN}{}^{(AB}\Psi^{CD)MN} = -\kappa\nabla^{(A}_{(A'}\nabla^{B}_{B')}T^{CD)A'B'}. \tag{110}$$

6 Concluding Remarks and Outlook

The curvature quantity Λ defined in Ref. [12] always obeys the relation

$$\frac{1}{8}R = \chi = 3\Lambda, \tag{111}$$

whence, by (10), the equality $\lambda = 2\chi$ holds when $T = 0$. In the absence of the cosmological constant, the number of degrees of freedom of a X-spinor for an empty spacetime is reduced to 10, and we correspondingly have the simplified expansion

$$\mathrm{X}_{ABCD} = \Psi_{ABCD}, \tag{112}$$

which still applies to the case of a trace-free $T_{\mu\nu}$ in the absence of λ. Under these circumstances, if the background spacetime also possesses conformal flatness, then the spinor X_{ABCD} must be taken as an identically vanishing object. For a conformally flat empty spacetime in the presence of the cosmological constant, we get from (61)

$$\mathrm{X}_{ABCD} = -\frac{1}{3}\lambda\varepsilon_{A(C}\varepsilon_{D)B}, \tag{113}$$

which reinstates the relation $\lambda = 2\chi$ through

$$\mathrm{X}_{AB}{}^{AB} = \lambda. \tag{114}$$

A noteworthy feature of our presentation is the relationship between the total symmetry of the source S_{ABCD} and the conservation law that is expressed as the covariant divergence freeness of $T_{\mu\nu}$. We saw that the contracted derivative (101) comes out at the intermediate stages of the expansions that yield Eq. (108),

but its vanishing has nevertheless to be thought of as a differential property related to Einstein's equations as well as to Eqs. (48) and (96). Thus, one could eventually be led to making the conservation statement upon keeping track of index configurations. Such results have provided us with a fresh insight into the situation concerning the derivation of generally relativistic wave equations as well as with a clear magnification of the spinor symmetries carried by the geometric structure of general relativity.

The source-free limiting case of Eqs. (109) and (110) is trivially defined by the statements

$$(\Box + \frac{1}{2}R)\Psi_{ABCD} - 6\Psi_{MN(AB}\Psi_{CD)}{}^{MN} = 0 \tag{115}$$

and

$$(\Box + \frac{1}{2}R)\Psi^{ABCD} - 6\Psi_{MN}{}^{(AB}\Psi^{CD)MN} = 0, \tag{116}$$

which control the propagation of gravitons in non-conformally flat vacuum spacetimes, in addition to coinciding formally with the statements formulated in Refs. [12,16]. We believe that further insights into the two-component theory of gravitons could be gained if the formulation just presented were extended to torsionful spacetime backgrounds.

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