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The BASICs of Branched Hamiltonians

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Abstract. Some examples of branched Hamiltonians are explored, as advocated by Shapere and Wilczek. These are actually cases of switchback potentials, albeit in momentum space, as previously analyzed for quasi-Hamiltonian dynamical systems in a classical context. A basic model, with a pair of Hamiltonian branches related by supersymmetry, is considered as an interesting illustration.

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Introduction to the problem

> *It is quite possible ... we may discover that in nature the relation of past and future is so intimate ... that no simple representation of a present may exist.*
>
> R.P. Feynman

In quantum mechanics

$$H = p^2 + V(x) \tag{1}$$

is neither more nor less difficult than

$$H = x^2 + V(p) \tag{2}$$

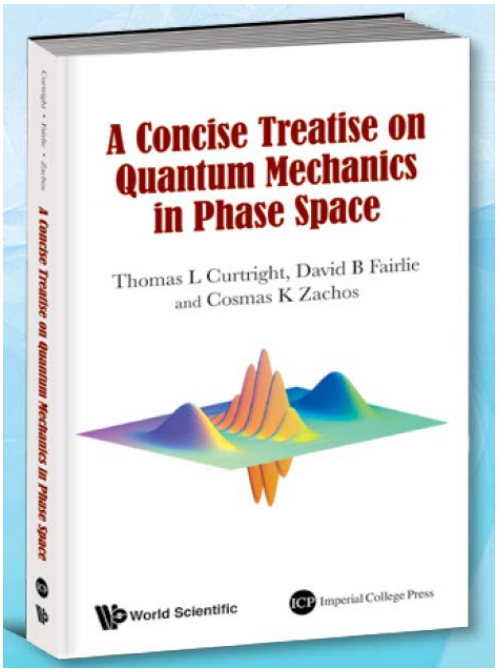
by reason of x, p duality, i.e. the Fourier transform:

$$\left.\begin{array}{c} \psi(x) \\ x \\ -i\hbar\,\partial/\partial x \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} \phi(p) \\ +i\hbar\,\partial/\partial p \\ p \end{array}\right. \tag{3}$$

This equivalence of (1) and (2) is manifest in the phase space formulation of quantum mechanics, as initiated by Wigner (1932) using his eponymous function:

$$f(x,p) = \frac{1}{\pi\hbar}\int dy \langle x+y|\rho|x-y\rangle \exp(-2ipy/\hbar)$$
$$= \frac{1}{\pi\hbar}\int dk \langle p+k|\rho|p-k\rangle \exp(+2ixk/\hbar)\,, \qquad (4)$$

where x and p are on an equal footing, and where even more general $H(x,p)$ can be considered. Please see our new book on the subject ...



Even in *classical* Hamiltonian mechanics, (1) and (2) are equivalent under a classical canonical transformation on phase space:

$$(x,p) \Longleftrightarrow (p,-x)\,. \qquad (5)$$

But upon transition to Lagrangian mechanics, the equivalence between the two theories becomes obscure.

Here is the issue: A *Legendre transformation* from (x,p,H) to (x,v,L) is complicated for *non-convex* $V(p)$. The resulting L is multi-valued, in general, with several branches.

Alternatively, if you are like Feynman in his youth, and keen to proceed from the start with a given single-valued $L(x,v)$, then you too will face similar complications if you are dealing with

$$L = x^2 - V(v) \qquad (6)$$

instead of the usual

$$L = v^2 - V(x)\,. \tag{7}$$

If you construct the Hamiltonian for (6) by Legendre transformation, in general you will encounter multi-valued-ness. (Shapere and Wilczek, 2012)

If $V(v)$ is non-convex, then the Legendre transformation $v \Longleftrightarrow p$ gives a multi-valued Hamiltonian, i.e. several branches for $H(x,p)$.

Again, the same issue: Starting from single-valued $H(x,p)$ or starting from single-valued $L(x,v)$ — either way — if the p or v dependence is non-convex then multi-valued, branched functions will arise upon Legendre transforming between Hamiltonian and Lagrangian formulations.

Let's consider some examples, say, one that I just pulled *out of my hat* ...

Miami Fedora Kinetic Energy

In dimensionless variables, define

$$\begin{aligned} L(x,v) &= -\exp(-v^2/2) + 1 - V(x) \\ &= \frac{1}{2}v^2 - V\,(x) + O(v^4)\,, \end{aligned} \tag{8}$$

e.g. $V(x) = 1 + x^2$ gives classical Euler-Lagrange equations

$$\begin{aligned} \frac{dv}{dt} &= \frac{-2x}{(1-v^2)}\exp(v^2/2) \\ &= -2x + O(v^2)\,. \end{aligned} \tag{9}$$

The RHS is a restorative force so long as $v^2 < 1$. For larger speeds the solutions explode.

The energy in terms of x and v is

$$E = (1+v^2)e^{-v^2/2} + V(x) - 1\,. \tag{10}$$

This is conserved, given the E-L equations, and so it gives the trajectories as constant E curves on the (x, v) plane (for sample trajectories, see [16]).

I will refer to the v-dependent terms in E as the "Miami fedora" kinetic energy for obvious reasons.

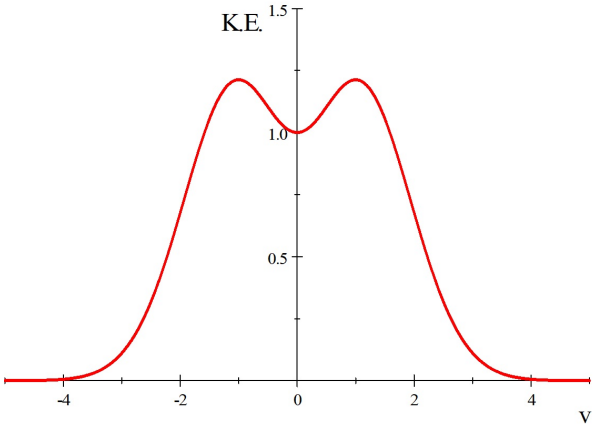


Figure 1. Miami fedora kinetic energy $(1+v^2)e^{-v^2/2} = 1 + \frac{1}{2}v^2 + O(v^4)$.

E.g. $V = 1 + x^2$ gives $x(v)$:

$$x = \pm\sqrt{E - K.E.} = \pm\sqrt{E - (1+v^2)e^{-v^2/2}}\,. \tag{11}$$

Or take $E = x^2 + (1+v^2)e^{-v^2/2}$, and solve for $v\,(x)$. The solution is:

$$v = \pm\sqrt{-1 - 2\,\mathrm{LambertW}\left(\frac{1}{2\sqrt{e}}(x^2 - E)\right)}\,. \tag{12}$$

For real v it is necessary here for the Lambert function, with negative argument, to return negative values. That is to say, either the principal branch, $\mathrm{LambertW}(0, z)$ or the lower branch $\mathrm{LambertW}(-1, z)$ with $-1/e \le z \le 0$.

So, $v\,(x)$ is multi-valued due to the different $\sqrt{\cdots}$ and $\mathrm{LambertW}\,(\cdots)$ branches.

Lambert function refresher course: The solution of

$$ye^y = z \tag{13}$$

is

$$y(z) = \begin{cases} \mathrm{LambertW}(k, z) \mid k \in \mathbb{Z} & \text{if} \quad z \neq 0\,, \\ 0 & \text{if} \quad z = 0\,, \end{cases} \tag{14}$$

where $k = 0$ and $k = -1$ give the two real branches as shown in Figure 2.

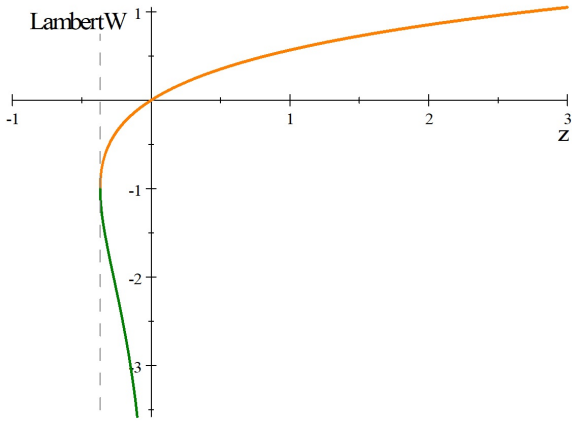


Figure 2. $\mathrm{LambertW}(0,z)$ and $\mathrm{LambertW}(-1,z)$ in orange and green, resp.

Legendre Transform to Obtain the Hamiltonian for the Model

The velocity dependent term is a union of three convex functions, defined on $v \in [-\infty,-1]$, $[-1,1]$, and $[1,\infty]$ (See Figure 3).

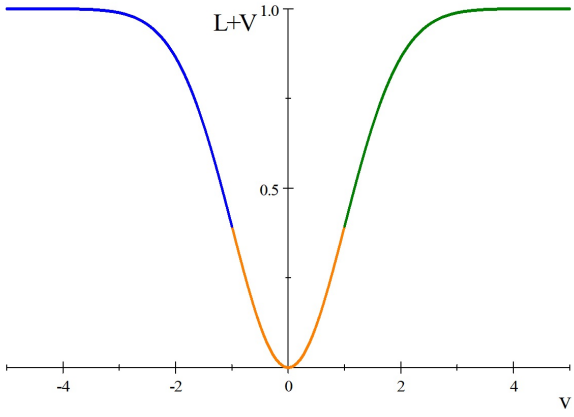


Figure 3. The model given by $L(x,v)+V(x)=1-\exp(-v^2/2)$.

As a consequence, the Hamiltonian will be multi-valued.

$$H = vp - L\,, \tag{15}$$

$$p = \frac{\partial L}{\partial v} = v\exp(-v^2/2)\,. \tag{16}$$

So as a function of v

$$H(x,v) = (1+v^2)\exp(-v^2/2) + V(x) - 1\,, \tag{17}$$

which we recognize as the previous E, of course. But we want $H(x,p)$. So we need $v(p)$.

The velocity as function of p again involves a Lambert function, as well as both branches of a $\sqrt{\cdots}$.

$$v = \pm\sqrt{-\,\mathrm{LambertW}(-p^2)}\,. \tag{18}$$

Again, for real v, negative values are required for $\mathrm{LambertW}$, so either the principal branch, $\mathrm{LambertW}(0,z)$ or the lower branch $\mathrm{LambertW}(-1,z)$ will do, with $-1/e \le -p^2 \le 0$. That is to say, the momentum lies in the finite interval

$$-\frac{1}{\sqrt{e}} \le p \le \frac{1}{\sqrt{e}}\,. \tag{19}$$

The result for $H(x,p)$ is multi-valued on this momentum interval, because of the square root and the Lambert function branches (See Figure 4)

$$H = \pm p\left(\sqrt{-\,\mathrm{LambertW}(-p^2)} + \frac{1}{\sqrt{-\,\mathrm{LambertW}(-p^2)}}\right) + V(x) - 1. \tag{20}$$

Note $\exp(-v^2/2) = p/v$.

$H(x,p)$ may be thought of as the union of three convex functions of p: H_-, H_0, and H_+ for $p \in [-1/\sqrt{e}, 0]$, $[-1/\sqrt{e}, 1/\sqrt{e}]$, and $[0, 1/\sqrt{e}]$, as displayed

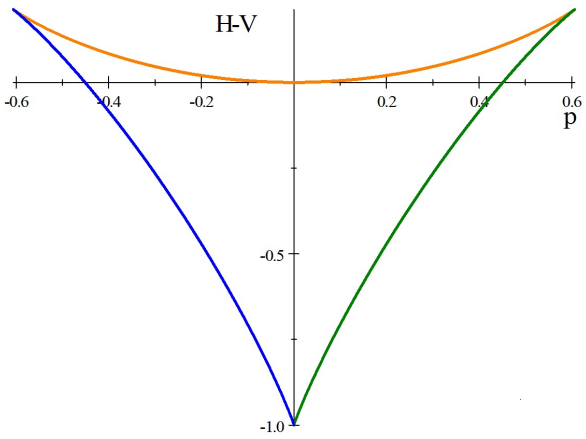


Figure 4. The real branches of $H - V$ versus $p \in [-1/\sqrt{e}, 1/\sqrt{e}] \approx [-0.607, 0.607]$.

above in blue, orange, and green, respectively. Classically, a particle switches H during the course of its trajectory. Different branches of H govern the motion at different times.

In a previous context (2010) — a study of the evolution of chaotic dynamical systems — Zachos and I found the problem was mathematically equivalent to particle motion in which the Hamiltonian (the potential $V(x)$, actually) switched when the particle encountered turning points, with continuous evolution of $x(t)$, $v(t)$, and $p(t)$ at the switching points. (Please see the literature cited below in the references.) The present example illustrates the *same switchback effects*, only here it is the momentum dependent part of H that switches.

For quantum mechanics, the Hamiltonian to be used in the Schrödinger equation also switches. The only additional issue is what are the boundary conditions on the wave function where the switches occur. Shapere and Wilczek impose conditions so that the probability current is conserved.

To shed light on such Hamiltonian switching quantum systems, I will *switch* gears from the Florida hat model to a more tractable example ...

Supersymmetric QM

Consider the single-valued real function, for $-\infty \leq v \leq +\infty$ (Figure 5),

$$L = C\,(v-1)^{1/3} - V(x) \tag{21}$$

where $C = 3/\sqrt[3]{16} \approx 1.19$. Take real roots on the real v axis.

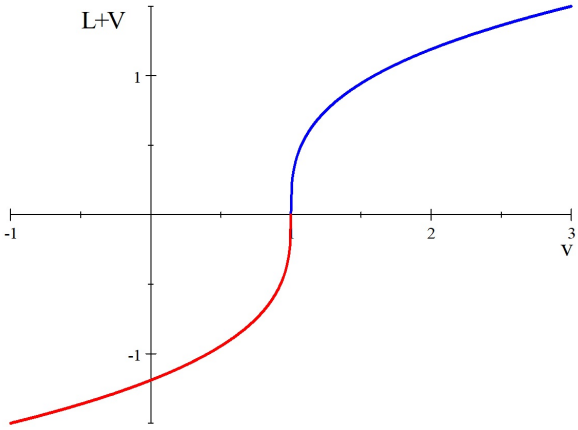


Figure 5. $L + V = C(v-1)^{\frac{1}{3}}$.

Proceeding with the Legendre transform

$$p = \frac{\partial L}{\partial v} \quad \Rightarrow \quad v_{\pm}(p) \equiv 1 \mp \frac{1}{4}\left(\frac{1}{\sqrt{p}}\right)^3 \tag{22}$$

and the Hamiltonian is ... double valued

$$H_{\pm} = p \pm \frac{1}{2\sqrt{p}} + V(x)\,. \tag{23}$$

From the shape of $(L+V)(v)$ curve, note that $p \geq 0$ (Figure 6).

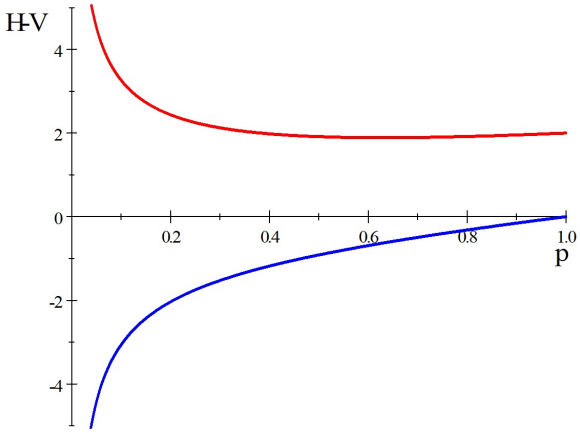


Figure 6. $H_{\pm} - V(x) = p \pm \frac{1}{2\sqrt{p}}$ in red/blue. There is a cusp at $p = \infty$.

Following the suggestions of Shapere and Wilczek, we define the associated *quantum* theory with $p \geq 0$ as a restriction, with various boundary conditions imposed on the wave functions, $\psi(p)$, at $p = 0$, such that there is no probability flow to negative p.

When the potential $V(x)$ is harmonic, this is a *supersymmetric* quantum mechanical system when viewed in momentum space.

$$V(x) = x^2 \underset{\text{QM in } p \text{ space}}{\longmapsto} -\frac{d^2}{dp^2}\,. \tag{24}$$

Quantum features: The momentum space supersymmetric pair of QM Hamiltonian operators for this case is therefore expressible in the standard form

$$H_{\pm} = -\frac{d^2}{dp^2} + w_0^2(p) \pm w_0'(p) = \Big(\frac{d}{dp} \pm w_0(p)\Big)\Big(-\frac{d}{dp} \pm w_0(p)\Big), \tag{25}$$

$$w_0(p) = \sqrt{p}\,. \tag{26}$$

This has the feature that the true — square-integrable — ground state of the system is non-vanishing for only one of the branches, namely, H_-.

As an algebraic system, for $p \geq 0$, the two Hamiltonians are related in a familiar fashion by

$$\begin{aligned} H_- &= a^\dagger a \,, \\ H_+ &= a a^\dagger \,, \end{aligned} \tag{27}$$

$$H_+ = H_- + [a, a^\dagger] \,, \tag{28}$$

$$a = \frac{d}{dp} + \sqrt{p} \,, \quad a^\dagger = -\frac{d}{dp} + \sqrt{p} \,, \quad [a, a^\dagger] = \frac{1}{\sqrt{p}} \,. \tag{29}$$

Obviously, either energy spectrum is non-negative.

The zero-energy ground state of H_- is given by

$$a\psi_0(p) = 0 \,, \quad \psi_0(p) = N_0 \exp\left(-\frac{2}{3}p^{3/2}\right) \,, \tag{30}$$

$N_0 = 6^{1/6}/\sqrt{\Gamma\left(\frac{2}{3}\right)} \approx 1.16$ ensures that $\int_0^\infty |\psi_0(p)|^2 dp = 1$. The ground state obeys the boundary condition $\psi_0'(0) = 0$.

On the other hand, the zero-energy state for H_+, namely, $\phi(x) = \exp\left(+\frac{2}{3}p^{3/2}\right)$, is not admissible, because it has infinite norm.

The higher energy states are degenerate, with $H_\pm \psi^{(\pm)} = E\psi^{(\pm)}$ eigenstates for $E > 0$ mutually related by

$$\psi_E^{(+)} = \frac{1}{\sqrt{E}}\, a\psi_E^{(-)} \,, \quad \psi_E^{(-)} = \frac{1}{\sqrt{E}}\, a^\dagger \psi_E^{(+)} \,, \tag{31}$$

so as to have equal norms. In particular the first excited state for H_- is degenerate with the lowest energy state for H_+, with $E_1 = 1.89379$, as determined by numerical analysis.

The degenerate $H_\pm$ eigenfunctions obey different boundary conditions at $p = 0$. If one is Dirichlet, the other is Neumann. For example, the first H_- excited state and its degenerate H_+ partner eigenstate satisfy

$$\psi_{E_1}^{(-)}\Big|_{p=0} = 0 = d\psi_{E_1}^{(+)}/dp\Big|_{p=0} \,,$$

while for the next excited states,

$$d\psi_{E_2}^{(-)}/dp\Big|_{p=0} = 0 = \psi_{E_2}^{(+)}\Big|_{p=0} \,, \quad \text{etc.}$$

All this conforms with *well-known* expectations for general supersymmetric QM, but the common *single-component* L underlying both branches of the Hamiltonian, in this case, is a new observation, so far as I know.

The flipping of the boundary conditions actually has a practical benefit due to the $1/\sqrt{p}$ singularity in both $H_\pm$: It is more straightforward to perform an accurate numerical computation of the energy eigenvalue using the boundary condition $\psi_E(0) = 0 \neq \psi'_E(0)$ than it is using the condition $\psi_E(0) \neq 0 = \psi'_E(0)$. The degeneracy of the eigenfunctions permits one to always choose the $\psi_E(0) = 0$ condition, along with the corresponding H_+ or H_-.

These higher energy states may be thought of as a single nontrivial state defined on a unified covering space — a double covering of the half-line $\mathbb{R}_+$ by $\mathbb{R}$ — obtained by unfolding the two Hamiltonian branches to obtain a single H globally defined on $\mathbb{R}$. However, as is clear from the preceding discussion, the true ground state of the system is $\psi_0(p) \cup 0$ on the unfolded space. The latter, somewhat unusual feature is possible because the two Hamiltonians on the half-lines join together in a cusp at $p = \infty$, where ψ_0 and all its derivatives vanish. So too vanish all the higher $\psi_E^{(\pm)}$ and all their derivatives at $p = \infty$.

For this reason, it would be excusable not to have thought of the degenerate eigenstates on the half-line as two branches of a single function. However, the unified picture provided by joining them together on a covering real line, with Neumann and Dirichlet boundary conditions at opposite ends, is a more compelling point of view, in our opinion. Perhaps more importantly, this omniscient view of the system becomes natural when the common Lagrangian underpinning both $H_\pm$ is considered.

Classical features: It is also instructive to survey essential features of the classical trajectories for the model. But time does not permit me to do this here. Please see my paper with Cosmas Zachos.

Discussion

As emphasized by Shapere and Wilczek, "many worlds" systems with branched Hamiltonians are by no means rare, in theory. Here, I have displayed some simple unified Lagrangian prototype systems which, by virtue of non-convexity in their velocity dependence, branch into double-valued (but still self-adjoint) Hamiltonians.

I have surveyed the spectral and boundary conditions involved for a supersymmetric model, in a uniform framework, by utilizing the eigenstate-linking "supercharge" ladder operators. These particular branched Hamiltonians — although governing "two worlds" — are nevertheless paired into a uniform isospectral system, in the very same Hilbert space.

Yet they are inexorably separated, in some analogy to fermionic and bosonic sectors, as the respective dynamical intervals only connect at $p = \infty$. In this respect, this particular supersymmetric system differs from more typical constructions given by Shapere and Wilczek, which exhibit similar operator branching structures but connect for finite p.

However, in addition, I have outlined another model whose branches lie on a compact, closed momentum manifold with coalescing cusps at finite p. The quantum features of this other model are in line with the cases considered by Shapere and Wilczek.

While only double-valued, H for the Miami fedora model is clearly the union of *three* convex functions, defined on three overlapping momentum intervals: H_-, H_0, and H_+ for $p \in [-1/\sqrt{e}, 0]$, $[-1/\sqrt{e}, 1/\sqrt{e}]$, and $[0, 1/\sqrt{e}]$, as displayed in a previous Figure in blue, orange, and green, respectively.

This unified 3-fold structure brings to mind some previous theories exhibiting triality, as discussed by Shankar (1981). However, to my knowledge the gaussian model shows no compelling signs of supersymmetry. Still, it would be quite interesting to find a simple, three-Hamiltonian, single-particle quantum system, based on a single unifying Lagrangian, that could be partitioned into pairs of supersymmetric Hamiltonians, with state-linking operators of the type analyzed above.

Acknowledgements

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