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Notes on Vortices and String/Gauge Theory Correspondence

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Abstract. In this notes we consider relations between strings, vortices, and 2D gauge theories. The framework we study these relations is string/gauge theory dualities. We show that certain quantities satisfy Toda equations. Next, we show that the brane construction of non-Abelian cortices can be embedded into ABJM theory and thus, it contains non-Abelian vortex solutions.

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1 Introduction

The unified understanding of Nature has been attracting the efforts of physicist's community for about half a century. The appealing idea of unification of all the fundamental interactions is a source of many speculations and often provokes scientific adventures which leave deep traces in our knowledge of the world. The quest for the true unified theory describing Nature is related, in last decades, in one or another way to string theory. In this respect, it is a challenging task to find the right place of the pieces which we know well from different theories in the puzzle of the string world. The other direction, the so-called top-bottom approach is equally important and frequently uncovers unexpected features of well known theories. In these notes we will take a look from both sides of this conjectured duality on the example of vortices and strings – a topic by far modest compared to the general conceptual issues but which may lead to far reaching conclusions.

The most fascinating direction of study of the relation between strings and gauge theories is the dual nature between the two theories. Actually the contemporary understanding of dualities originates from string theory offering invaluable way to understand the non-perturbative physics [1]. The main example of these phenomena is AdS/CFT [2], or more generally holographic correspondence which schematically works as shown in Figure 1.

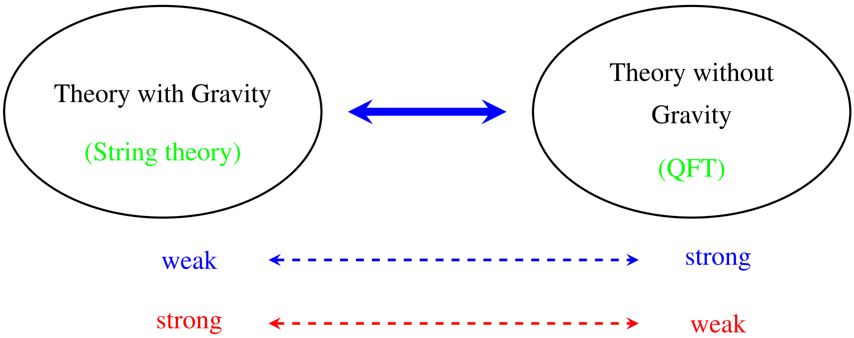


Figure 1. The duality between strings and gauge theories.

Let us briefly describe the string construction of gauge theories. As it is well known, the closed string sector determines the string background, or in other words, it determines the (super) gravity approximation of the theory. When open string sector is included, the picture changes. It is due to boundary conditions we must furnish the theory with. The imposition of boundary conditions dictates the introduction of new non-perturbative objects – D_p branes. The later are $p+1$-dimensional objects which account for giving Dirichlet boundary condition to the string ends. Let us now consider intersecting D_p branes. The so-called Chan-Paton factors enumerating string ends get promoted to dynamical degrees of freedom on the D_p branes. At the intersection the strings are viewed as point particles transforming under certain gauge group. We can summarize this picture as in Figure 2:

- The string endpoints on the same D_p branes transform under adjoint representation of the gauge group. The large number of these branes gives the background geometry.
- The string endpoints ending on different D_p–$D_{p'}$ branes transform in the fundamental. This is the way we introduce flavors in the theory.

The dualities we have briefly discussed above have many different faces depending on the concrete construction. We will focus only on the dualities, which involve vortex solutions and play important role in understanding of the two sides on the conjectured duality, see for instance the reviews [3] and [4] and references therein. First we will say a few words about the motivations leading to this problem. There are several important topics in high energy physics, which are deeply related to the vortex theory and its numerous applications. Some immediate particularly hot topics (picked almost randomly) are the following:

- Let us consider a system in an external magnetic field $\vec{B}$. The external magnetic field induces a chiral current

$$\vec{J}_{cme} \sim \mu_5 \vec{B}$$

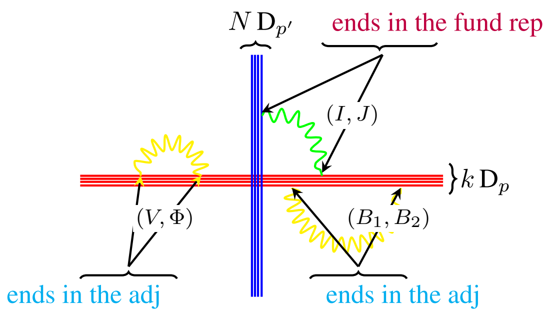


Figure 2. The transformation of string endpoints as viewed from the intersection of D_p branes.

where μ_5 is the axial chemical potential. Applied to holographic theories in string theory, the external "magnetic field" excites a certain form on particular D-brane. This results in chiral symmetry breaking and the effect is called holographic chiral magnetic effect.
In the very same way one can consider a vortex generated by a vorticity vector $\vec{\omega}$. The induced current is

$$\vec{J}_{cve} \sim \mu_5 \mu \, \vec{\omega} \,,$$

where μ is the vector chemical potential. The effect is called chiral vortical effect. In holographic correspondence it is called holographic chiral vortical effect respectively.

- In a given theory we usually want to calculate the partition function. This however is, as a rule, a hard task. Nevertheless, it is very helpful to calculate particular indices which capture particular very important properties of the theory. Such indices are the Witten index, superconformal index, etc. The presence of topological solitons frequently is very helpful in doing that.
- In 3D higher spin theories we have black hole solutions which correspond in the gravity/gauge theory correspondence to theories at finite temperature. Recently it has been demonstrated that the chemical potentials can be identified with the conserved charges of the mKdV (modified Korteveg-de Vries) hierarchy. Therefore, there is a potential relation to soliton solutions and their classification in terms of hierarchies.
- Finally, as Wald has shown, the entanglement entropy can be thought of as a conserved charge. In the case of integrable structures, it is tempting to look for identification of certain charges of the model with the entanglement entropy. It is more important for our purposes that it can be easily accommodated in the holographic correspondence if true.

A lot of other issues can be added to the list of important topics and potential directions of development, but now we will stop at this point.

In this short notes we will try to give, certainly not complete but hopefully convincing arguments concerning some relations between string theory/gauge theory dualities involving vortices.

2 Non-Abelian Vortices

The Abelian vortices are known for a long time and there is vast of literature devoted to their properties and applications. Here we will skip the description of these vortices and will concentrate on the non-Abelian ones. The non-Abelian vortices were introduced in the second half of 90's by Nekrasov and Shifman. They are constructed as follows.

The Field content:

- Gauge fields A_μ, $\mu = 0, 1, \ldots, d-1$
- Two Higgs fields: $N \times N$ matrices H^1, H^2
- Adjoint scalars coming from dimensional reduction ϕ^r, $r = 1, \ldots, 6-d$.

The Lagrangian density. We give below the lagrangian in 6D. The other cases can be obtained by trivial dimensional reduction[1].

$$\mathcal{L}_{6d} = \mathrm{tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + D^\mu H^i (D_\mu H^i)^\dagger\right] - V, \tag{1}$$

where (we include the covariant derivative for eventual scalars ϕ^r coming from the dimensional reduction)

$$\begin{aligned} &D_\mu \phi^r = \partial_\mu \phi^r + i[A_\mu, \phi^r], \quad D_\mu H = (\partial_\mu + iA_\mu)H, \\ &F_{\mu\nu} = -i[D_\mu, D_\nu]. \end{aligned} \tag{2}$$

The potential V is given by

$$V = \frac{g^2}{4}\mathrm{tr}\left[\left(H^1 H^{1\dagger} - H^2 H^{2\dagger} - c\mathbf{1}\right)^2 + 4H^2 H^{1\dagger} H^1 H^{2\dagger}\right]. \tag{3}$$

• Dimensional reduction ⇒ additional terms:

$$\frac{1}{2g^2}[\phi^r, \phi^s]^2 + (H^2 H^{2\dagger} + H^1 H^{1\dagger})\phi^r \phi^r. \tag{4}$$

The triplet of Fayet-Iliopoulos parameters is chosen to the third direction: $(0, 0, c)$.

[1] The gauge field A_μ, $\mu = 0, 1, 2$ depends only on 3D coordinates. The other components of the connection become scalars ϕ^r depending also on 3D coordinates.

• The SUSY requires vanishing of the vacuum energy. This means that the following equations must be satisfied:

$$D_1 H + i D_2 H = 0, \tag{5}$$

$$F_{12} = \frac{g^2}{2}(HH^\dagger - c\mathbf{1}_N). \tag{6}$$

The additional terms coming from the dimensional reduction force $\phi^r = 0$, $r = 1, 2, 3$.

Symmetries:

- Gauge symmetry: $U(N_C)$
- Flavor symmetry: $SU(N_F)$.

The Higgs fields: H^1 transforms as $(\mathbf{N}, \bar{\mathbf{N}})$ while H^2 transforms as $(\bar{\mathbf{N}}, \mathbf{N})$. The explicit matrix structure of these field is as follows:

$$H^1 \equiv H^{1a}_{\ \ i},\ H^2 \equiv H^{2i}_{\ \ a}, \quad a = 1, \dots, N_C, \quad , i = 1, \dots, N_F$$

where a are $U(N_C)$ index and i is $SU(N_F)$ index.

In either dimension, the vacuum is the so-called color-flavor locking phase, i.e. the ground state develops a gap. This phase is characterized by

$$H^1 = \sqrt{c}\mathbf{1}_N, \quad \& \quad H^2 = 0. \tag{7}$$

In this phase the symmetry is broken

$$U(N_C) \times SU(N_F) \xrightarrow{c>0} SU(N)_{(C+F)}. \tag{8}$$

The symmetry is further broken by the presence of a vortex:

$$SU(N)_{(C+F)} \xrightarrow{\text{vortex}} U(1)^N.$$

Minimizing the energy density, one we conclude that the Bogomol'nyi bound is saturated iff

$$D_1 H + i D_2 H = 0, \tag{9}$$

$$F_{12} = \frac{g^2}{2}(HH^\dagger - c\mathbf{1}_N).. \tag{10}$$

The tension of the vortex string measures the winding number k of the $U(1)$ part of the broken $U(N_C)$ gauge symmetry

$$T = -c \int dx \mathrm{tr} F_{12} = 2\pi c k, \quad k \in \mathbb{Z}_{\geq 0}. \tag{11}$$

To solve (9) we make an ansatz

$$H = S^{-1}(z, \bar{z}) H_0(z), \tag{12}$$

where $S^{-1}(z, \bar{z}) \in GL(N_c, \mathbb{C})$ and $H_0(z)$ is an arbitrary. Substituting (12) into (9), we find

$$A_1 + iA_2 = -2iS^{-1}(z, \bar{z})\partial_{\bar{z}} S(z, \bar{z}). \tag{13}$$

The matrix $\mathbf{H_0}(z, \bar{z})$ is called *moduli matrix* and is introduced first in [5]. The asymptotic behavior and the vorticity k require the relation

$$T = 2\pi ck = -\frac{c}{2} i \oint dz \partial_z \log \det(H_0 H_o^\dagger) + c.c. \tag{14}$$

Moduli space. The moduli space has two types of moduli: position moduli and orientation moduli. For simplicity we will consider only the $N_c = N_F$ moduli space.

The position moduli z_i. The vorticity condition (14) can be conveniently written as

$$k = \frac{1}{2\pi} \Im \oint dz \partial \log(\det H_0), \tag{15}$$

which leads to boundary conditions for H_0 on S'_∞:

$$\det(H_0) \sim z^k, \quad \text{for } z \to \infty. \tag{16}$$

Therefore, H_0 has k zeroes, $z = z_i,\ i = 1, \ldots, k$, which are the positions of the vortices. We will assume for now that all the roots are different, i.e. the vortices are separated

$$P(z) = \det(H_0) = \prod_{i=1}^{k} (z - z_i). \tag{17}$$

The orientation moduli $\vec{\phi}$. The orientation moduli are determined by the equation

$$H_0(z_i)\vec{\phi}_i = 0 \quad \Longleftrightarrow \quad H(z = z_i, \bar{z} = \bar{z}_i)\vec{\phi} = 0. \tag{18}$$

The moduli space for $k = 1$ vortices is: $\mathcal{M}_{N,k=1} = \mathbb{C} \times \mathbb{CP}^{N-1}$, and for l copies of them looks like

In the case of arbitrary k the moduli space is more complicated. For instance in the case of $N_C = N_F$

$$\mathcal{M}_{N,k} = \frac{\{H_0(z) \,|\, H_0(z) \in M_N,\ \deg(\det(H_0)) = k\}}{\{V(z) \,|\, \tilde{V}(z) \in M_N,\ \det \tilde{V}(z) = 1\}}, \tag{19}$$

where $M_N = N \times N$ *are matrices of polynomials of* z.

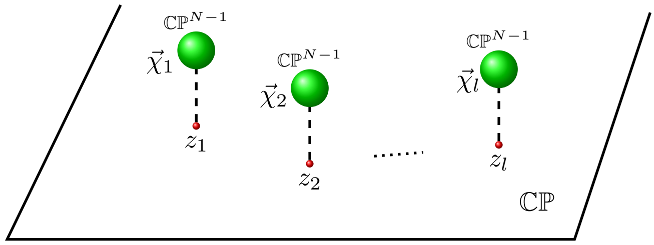


Figure 3. The moduli space of $k = 1$ vortex.

Using appropriate transformations, one can put $H_0(z)$ into upper-triangular form

$$H_0(z) = \begin{pmatrix} P_1(z) & R_{2,1}(z) & R_{3,1}(z) & \cdots & R_{N-1,1}(z) & R_{N,1}(z) \\ 0 & P_2(z) & R_{3,2}(z) & \cdots & R_{N-1,2}(z) & R_{N,2}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & P_N(z) \end{pmatrix}, \quad (20)$$

where

$$P_r(z) = \prod_{i=1}^{k_r}(z - z_{r,i}),\ k = \sum k_r, \quad \& \quad \deg(R_{r,m}) < k_m = \deg(P_m(z)). \quad (21)$$

In general the moduli matrix is assembled from the N_F hypers in the fundamental in gauge theory with N_C color symmetry as

$$H^i = \begin{pmatrix} H^{i11} & H^{i12} & \cdots & H^{i1N_F} \\ H^{i21} & H^{i22} & \cdots & H^{i2N_F} \\ \vdots & \vdots & \vdots & \vdots \\ H^{iN_C1} & H^{iN_C2} & \cdots & H^{iN_CN_F}, \end{pmatrix}, \quad (22)$$

where the matrix fields H^i satisfy BPS conditions

$$H^1 H^{1\dagger} - H^2 H^{2\dagger} = c\mathbf{1}, \quad H^2 H^{1\dagger} = 0. \quad (23)$$

One can consider also the inclusion of fermions. This part is not essential for our discussion and we quote the answer for completeness. As in the bosonic case, one can conclude that the fermionic directions of the vortex moduli space are

$$\{\tilde{\psi}_{0-}(\bar{z})\} \cong \mathbb{C}^{\bar{N}} \otimes (P_{k_1}[\bar{z}] \oplus \cdots \oplus P_{k_N}[\bar{z}])\,. \quad (24)$$

3 Relation to Toda Theories

3.1 Abelian case

From now on we will use the following notations:

- ϕ is a complex smooth scalar field on $\mathbb{R}^2$ ($\equiv H^1$ above).
- $A_i, \;\; i = 1, 2$ is a smooth real vector field.

We assume that the above two fields satisfy

$$D_{\pm}\phi := (\partial_1 \pm i\partial_2)\phi - i(A_1 \pm iA_2)\phi = 0.$$

For $\omega \in \mathbb{R}^2$ smooth and real, we have the invariance

$$\phi \;\rightarrow\; e^{\omega}\phi; \quad A_i \;\rightarrow\; A_i + \partial_i\omega.$$

Thus, we use the freedom to make the choice:

- We assume that $\mathcal{A}$ is specified according to the Coulomb gauge, i.e. it is divergence-free
$$\partial_1 A_1 + \partial_2 A_2 = 0. \tag{25}$$
- We choose a real function, ξ, satisfying

$$\nabla\xi = \pm(-A_2, A_1), \tag{26}$$

$$\psi := e^{-\xi}\phi \quad \Rightarrow \quad (\partial_1 \pm i\partial_2)\psi = 0. \tag{27}$$

To solve the theory we employ the following strategy. Let $\{z_1, \ldots, z_N\}$ be the set of zeroes of ϕ (with multiplicities). It is useful for our purposes to explicitly separate the zeroes

$$\hat{\phi}(z, \bar{z}) := e^{\xi(z,\bar{z})} \prod_{j=1}^{N} (z - z_j)\, \hat{h}(z), \tag{28}$$

where $\hat{h}(z)$ is non-vanishing (and $\hat{h}^{-1}$ exists!)

Now we can change the gauge as

$$\hat{\phi} \;\longrightarrow\; |\hat{h}|\hat{h}^{-1}\hat{\phi}. \tag{29}$$

Then one can define the field ϕ using ξ and $\hat{h}$ as

$$\phi(z) = |\phi(z)| e^{\pm i \sum\limits_{j=1}^{N} \varphi_j}, \quad |\phi(z)| = e^{\xi} |\hat{h}(z)| \prod_{j=1}^{N} |z - z_j|,$$

$$|\hat{h}| = |\det\, \hat{h}(z)|, \quad \varphi_j = \mathrm{Arg}(z - z_j).$$

The field equation, namely $D_{\pm}\phi = 0$ becomes

$$\partial_2 \log|\phi| + \partial_1 \sum_{j=1}^{N} \varphi_j = \pm A_1,$$
$$\partial_1 \log|\phi| - \partial_2 \sum_{j=1}^{N} \varphi_j = \mp A_2. \tag{30}$$

We list the important properties of (30):

- The equations (30): smooth expressions for A_i in terms of gauge invariant quantity $|\phi|$!
- Having that $\nabla\xi = (-A_2, A_1)$, one easily finds that $\log|h|$ is a harmonic.
- For the Laplacian of $\log|\phi|$ we find

$$\begin{aligned}-\Delta \log|\phi|^2 &= -2\Delta\xi - 2\Delta \log|h|^2 - 4\pi \sum_{j=1}^{N} \delta(z - z_j)\\ &= \pm F_{12} - 4\pi \sum_{j=1}^{N} \delta(z - z_j).\end{aligned} \tag{31}$$

To make connection with Liouville theory let us define

$$\rho := \log|\phi|^2, \quad \& \text{ set of zeroes } \{z_1, \ldots, z_N\}. \tag{32}$$

Then the field $\phi(z)$ takes the form familiar from Liouville theory

$$\phi(z) = e^{\frac{1}{2}\rho(z) \pm i \sum\limits_{j=1}^{N} \varphi_j}. \tag{33}$$

The equations (30) in terms of ρ are

$$A_1 = \pm\left(\frac{1}{2}\partial_2\rho + \partial_1 \sum_{j=1}^{N} \varphi_j\right) \tag{34}$$

$$A_2 = \mp\left(\frac{1}{2}\partial_1\rho - \partial_2 \sum_{j=1}^{N} \varphi_j\right). \tag{35}$$

3.2 Non-Abelian case

Consider non-Abelian Higgs-CS theory defined with the combined Lagrangians

$$\begin{aligned}\mathcal{L}(\mathcal{A},\phi) &= k\mathcal{L}_{CS} + \mathrm{tr}\left[D_a\phi(D^a\phi)^\dagger\right] - V \\ \mathcal{A}_{CS} &= \frac{\epsilon^{\mu\nu\alpha}}{2}\mathrm{tr}\left(A_\mu\partial_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\right) \\ V &= \frac{1}{k^2}\left|[[\phi,\phi^\dagger],\phi] - v^2\phi\right|^2 .\end{aligned} \tag{36}$$

Expanding explicitly on the generators of the gauge group, $\phi = \phi^a E_a$, we find

$$\begin{aligned}&V = \frac{1}{k^2}\left|\phi^a(v^2 - C_{ba}|\phi^b|^2)\right|^2, \quad D_a D^a\phi = \frac{\partial V}{\partial\phi^\dagger} \\ &\frac{k}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} = -iJ^\mu, \qquad J^\mu = i\left([D^\mu\phi,\phi^\dagger] - [\phi,(D^\mu\phi)^\dagger]\right).\end{aligned}$$

The setup in the non-Abelian case uses

$$\mathcal{A} = A_\alpha dx^\alpha; \quad A_\alpha = -iA^a_\alpha T_a; \quad \phi = \phi^a E_a, \tag{37}$$

where T_a are the generators of the gauge group, E_a are the simple roots generators, $A^a_\alpha \in \mathbb{R}$, $\phi^a \in \mathbb{C}$, a set of zeroes of ϕ, $\{z_1,\ldots,z_{N_a}\}$, where $N_a \in \mathbb{N}$.

For the non-Abelian generalization

$$\phi^a := e^{\frac{1}{2}\rho_a \pm i\sum\limits_{j=1}^{N}\varphi^a_j}, \quad a = 1,\ldots,r; \;\; \varphi^a_j = \mathrm{Arg}(z - z^a_j).$$

$$\begin{aligned}C_{ba}A^b_1 &= \pm\left(\frac{1}{2}\partial_2\rho_a + \partial_1\sum_{j=1}^{N}\varphi^a_j\right) \\ C_{ba}A^b_2 &= \mp\left(\frac{1}{2}\partial_1\rho_a - \partial_2\sum_{j=1}^{N}\varphi^a_j\right) \\ C_{ab}A^b_0 &= \pm\frac{1}{k}\left(v^2 - C_{ba}e^{\rho_b}\right).\end{aligned}$$

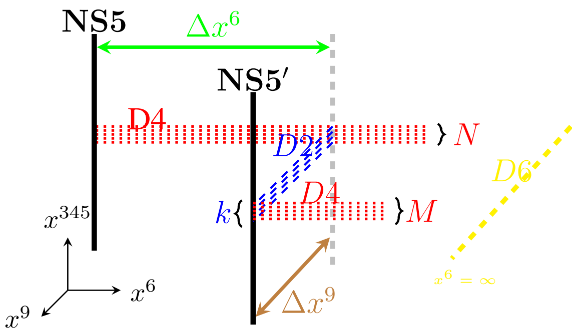
In these notations the self-dual vortex equations become

$$-\Delta\rho^a = \frac{4}{k^2}\left(v^2C_{ba}e^{\rho_b} - C_{ba}e^{\rho_b}C_{cb}e^{\rho_c}\right) - 4\pi\sum_{j=1}^{N}\delta(z - z^a_j).$$

Specializing to $SU(n+1)$ we get Toda lattice system coupled to $SU(n+1)$ Cartan matrix C_{ab}.

4 The Other Side of the Story

We turn now to the string/brane construction of the vortices. As was suggested by Hanany and Tong [6], the strings stretched between certain configuration of D_p branes gives on their intersection *exactly* the field content and field equations of vortices. For the case of NS5-D4-D2-D6 system, where sending D6 to infinity is accompanied with Hanany-Witten effect, the relevant configuration is given in Figure 4. The field content correspond to the scheme given in Figure 2 and described for instance by Hanany and Tong.



the gauge coupling g: $g = \frac{\Delta x^6 l_s}{2 g_s}$; FI parameter: $(c =) v^2 = \frac{\Delta x^9}{(2\pi)^3 g_s l_s^3}$

Figure 4. The brane construction of non-abelian vortices and the corresponding parameters (Hanany-Tong).

The alternative view we take here is based on the moduli matrix introduced above. As we already discussed, the entries of the moduli matrix are polynomials of complex variable of certain degree, see for instance (20). To employ the brane engineering of the vortex moduli space we have to establish its structure and introduce appropriate coordinates. First, let us remind that the moduli space of k-vortices in $U(N)$ gauge theory consists of $M_N = N \times N$ *matrices of polynomials of* z:

$$\mathcal{M}_{N,k} = \frac{\{H_0(z)\,|\,H_0(z) \in M_N,\ \deg(\det(H_0)) = k\}}{\{V(z)\,|\,\tilde{V}(z) \in M_N,\ \det \tilde{V}(z) = 1\}}. \tag{38}$$

As noted for instance in [5], to conveniently establish the global structure of the moduli space, it is convenient to introduce patches covering $\mathcal{M}_{N,k}$. We choose the following parametrization

$$\begin{aligned} (H_0)^r_s &= z^{k_r}\delta^r_s - T^r_s(z) \\ T^r_s(z) &= \sum_{n=1}^{k_s} (T_n)^r_s z^{n-1} \in \mathrm{Pol}(z; k_s) \end{aligned} \tag{39}$$

Therefore, the moduli parameters chosen as coordinates in the corresponding patch are

$$\{(T_{n_s})^r_s\}, \quad n_s = 1, \ldots, k_s;\ r = 1, \ldots, N_c.$$

The convenient choice of coordinates on a given patch is[2]

$$\xi_{A_s} = \det\left(H_0^{(\{A_s\})}(z)\right), \tag{40}$$

where

— the maximal degree of ξ_{A_s} is k_s ($\{A_{s_r}\}$ label the different vacua);
— $H_0^{(\{A_s\})}(z)$ is a $N_{k_s} \times N_{k_s}$ minor of H_0;
— the entries are ordered, $k_s > k_r$ for $s < r$.

We have focused here on the massive case of semi-local vortices. It is well known that in the massless case the theory acquires additional moduli, namely the size moduli. However, in the massive case here these additional moduli parameters are lifted and only several different species of ANO vortices remain. In fact, the massless vacuum manifold Gr_{N_F,N_C} reduces to the ${}_{N_F}C_{N_C}$ discrete vacua labeled as above by $(A_1, A_2 \cdots, A_{N_C}) \equiv (\{A_s\})$ by the non-degenerate masses. We remind that the positions of the vortices (or vev's of the fields) determine the positions of the branes, while the separation between the stacks on NS5 corresponds to masses. This description leads to the picture depicted in Figure 4 with the correct field content and the right non-abelian vortex theory.

Compactifying x^6 direction on a circle, D4 branes become D3, and placing N(p,q) branes instead of two NS5 branes we construct a quiver with a dual theory the so-called ABJM (Aharony-Bergman-Jafferis-Maldaceba) theory [9]. Next, it was shown in [8] that some quantities satisfy the Toda field equations. On other side, it was shown that the field structure on the intersection of these brane stacks gives multiplets and field theory of vortices [10] and deep connection with Toda theory was conjectured. Later in was proven in [11] that Abelian vortices in ABJM theory exist indeed. Here we gave additional arguments that not only Abelian vortices but non-Abelian ones are also presented in the theory. Thus, one can conclude that the non-Abelian vortices essentially participate in the celebrated holographic duality for this particular case.

5 Conclusions

The string/gauge theory duality is hot and rapidly developing area. It is the only reliable tool so far to study non-perturbative physics which is out of reach by other methods. Particular realization of this ideas is the so-called AdS/CFT correspondence achievements of which have been spread over large areas of

[2]The moduli space of vortices $\mathcal{M}_{N,k}$ is a Grassmaniann and is parameterized by Plücker coordinates ξ_{A_r}.

physics – particle physics, condensed matter physics, black holes, cosmology, even information theory, etc.

In this short notes we reported on the role of non-Abelian vortices in string/gauge theory duality. We consider the non-Abelian vortices from differen points of view. First, we give a standard description of the non-Abelian vortices. Then, we consider their relations to Toda field theory, which was anticipated for some time. Next we considered the constructions in terms of D_p branes which results in the non-Abelian vortex theory on the intersection of certain stacks of branes. Compactifying one of the direction along the D4 branes and replacing Hanany-Witten effect by substituting NS5 branes by $\mathrm{D}_{(p,q)}$ branes we end up with ABJM theory. Going back to brane construction of the vortices we realize that ABJM theory contains non-Abelian vortices. On other hand, the scalar field $\rho^a = \log|\phi^a|$ was shown to satisfy Toda equations.

There are many directions of development of ideas presented here. First, one can start considering non-Abelian vortices as defects in certain constructions of string/gauge theory dualities. Next, one can apply these findings to AdS/CMP (condensed matter physics). The Abelian vortices are well presented in QCD. Here there are many avenues of developments. As discussed in the Introduction, there are also many other issues related to finite temperature physics, fractional Hall effect, matrix models etc. We hope that our notes will be good motivation to go in some of these directions.

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