Anisotropic Dark Energy Model with a Special Form of Deceleration Parameter

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Abstract. Anisotropic Bianchi type-VI₀ universe with variable equation of state (EoS) parameter is obtained in a scalar-tensor theory of gravitation proposed by Saez and Ballester (Phys. Lett. A 113 (1986) 467). It is observed that the dark energy cosmological model always represents an accelerated and expanding universe and also consistent with the recent observations of type-Ia supernovae. The physical and kinematical properties of the universe have been discussed.

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1 Introduction

The cosmological observations of type Ia Supernovae (SNe Ia) indicate the accelerated expansion of the universe in the current era [1,2]. In addition, the combination of results from the large scale distribution of galaxies and the most precise data on the cosmic microwave background (CMB) from the Wilkinson Microwave Anisotropy Probe (WMAP) confirm such a cosmic acceleration [3,4]. The WMAP shows that Dark Energy (DE) occupies about 73% of the energy of our universe, and dark matter about 23%. The usual baryon matter, which can be described by our known particle theory, occupies only about 4% of the total energy of the universe. The acceleration is realized with negative pressure and positive energy density that violates the strong energy condition. This in turn gives reverse gravitational effect because of which the universe gets a jerk. This causes a transition from the decelerated phase to the accelerated phase [5]. The DE model has been characterized in a conventional manner by the EoS parameter \( \omega = p/\rho \), which is not necessarily constant, where \( \rho \) is the energy density and \( p \) is the fluid pressure [6]. The present data seem to slightly favor an evolving DE with EoS \( \omega < -1 \) around the present epoch and \( \omega > -1 \) in the recent past. Obviously, \( \omega \) cannot cross \(-1\) for quintessence or phantoms alone. In recent years
many authors [7-14] have obtained dark energy models in general relativity with variable EoS parameter.

In recent years there has been several modifications of general relativity. Note-worthy among them are scalar-tensor theories of gravitation proposed by Brans and Dicke [15] and Saez and Ballester [16]. Brans-Dicke theory introduces an additional scalar field φ besides the metric tensor \( g_{ij} \) and a dimensionless coupling constant \( \omega \). This theory goes to general relativity for large values of the coupling constant > 500. In Saez-Ballester scalar-tensor theory the metric is coupled with a dimensionless scalar field in a simple manner. This coupling gives a satisfactory description of the weak fields. Despite of the dimensionless character of the field an antigravity regime appears in this theory. Also, this theory suggests a possible way to solve missing matter problem in non-flat FRW cosmologies. Rao et al. [17] have discussed dark energy model of Bianchi type-I in Saez-Ballester scalar-tensor theory of gravitation with variable EoS parameter. Naidu et al. [18,19] have obtained Bianchi type-II and V dark energy models in the scalar-tensor theory of gravitation proposed by Saez and Ballester [16]. Bianchi type-III dark energy model with variable EoS parameter in Saez-Ballester scalar tensor theory of gravitation have been investigated by Naidu et al [20] with the help of the variation law for generalized Hubble’s parameter given by Bermann [21]. Spatially homogeneous Bianchi type-II, VIII & IX dark energy anisotropic as well as isotropic cosmological models with variable equation of state (EoS) parameter have been investigated by Rao et al. [22] in scalar tensor theory of gravitation proposed by Saez and Ballester [16]. Katore et al. [23] investigated homogeneous and anisotropic Kantowaski-Sachs space-time filled with Dark Energy in the Saez-Ballester theory.

Motivated by above research works, in the present paper, Bianchi type-VI\(_0\) cosmological models with anisotropic dark energy and special form of deceleration parameter have been studied in a scalar-tensor theory of gravitation proposed by Saez and Ballester [16]. The physical and kinematical properties of the universe have been discussed.

2 Metric and Its Field Equations

The anisotropic Bianchi type-VI\(_0\) metric is in the form

\[
\text{ds}^2 = -dt^2 + A^2 dx^2 + B^2 e^{-2qx} dy^2 + C^2 e^{2qx} dz^2,
\]

where \( A, B, C \) are the functions of cosmic time \( t \) only and \( q \) is the non zero constant.

The field equations given by Saez and Ballester [16] for the combined scalar and tensor fields are

\[
R_{ij} - \frac{1}{2} g_{ij} R - \omega \phi^n (\phi,\phi,_{ij} - \frac{1}{2} g_{ij} \phi,\phi,^k) = -T_{ij}
\]

(2)
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and the scalar field $\phi$ satisfies the equation

$$2\phi^n \phi_k^i + n\phi_{,k}\phi^k = 0.$$  \hspace{1cm} (3)

Here $\omega$ and $n$ are constant, $T_{ij}$ is the energy momentum tensor of the matter and comma and semicolon denote partial and covariant differentiation respectively.

Also

$$T_{i; i} = 0,$$  \hspace{1cm} (4)

is consequence of the field equations (2) and (3).

The energy momentum tensor of the fluid is taken as

$$T_i^j = \text{diag} [T_1^1, T_2^2, T_3^3, T_4^4].$$  \hspace{1cm} (5)

The simplest generalization of EoS parameter of perfect fluid is to determine it separately on each spatial axis by preserving diagonal form of the energy momentum tensor in a consistent way with the considered metric. Hence one can parameterize energy momentum tensor as follows:

$$T_i^j = \text{diag} [p_x, p_y, p_z, -\rho],$$ $$T_i^j = \text{diag} [w_x, w_y, w_z, -1] \rho,$$ $$T_i^j = \text{diag} [w, (w + \delta), (w + \eta), -1] \rho.$$  \hspace{1cm} (6)

Here $\rho$ is the energy density of the fluid, $p_x, p_y, p_z$ are the pressures and $w_x, w_y$ and $w_z$ are the directional EoS parameters along the $x, y,$ and $z$ axes respectively, $w$ is the deviation free EoS parameter of the fluid. We have parameterized the deviation from isotropy by setting $w_x = w$ and then introducing skew ness parameters $\delta, \eta,$ which are the deviations from $w$ along both $y$ and $z$ axes.

The Einstein field equations for metric (1) with the help of equations (2), (3) and (6) become (choosing $8\pi G = K = 1$)

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} + \frac{q^2}{A^2} - \frac{\omega}{2} \phi^n \phi^4 = \rho,$$  \hspace{1cm} (7)

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} - \frac{q^2}{A^2} - \frac{\omega}{2} \phi^n \phi^4 = -(w + \delta) \rho,$$  \hspace{1cm} (8)

$$\frac{B_{44}}{B} + \frac{A_{44}}{A} + \frac{B_4A_4}{BA} - \frac{q^2}{A^2} - \frac{\omega}{2} \phi^n \phi^4 = -(w + \eta) \rho,$$  \hspace{1cm} (9)

$$\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC} - \frac{q^2}{A^2} + \frac{\omega}{2} \phi^n \phi^4 = \rho,$$  \hspace{1cm} (10)

$$\frac{B_4}{B} - \frac{C_4}{C} = 0,$$  \hspace{1cm} (11)

$$\phi_{44} + \phi_4 \left( \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right) + \frac{n}{2} \phi^3 \phi_{4} = 0,$$  \hspace{1cm} (12)
where the subscript 4 denotes differentiation with respect to $t$.

The deceleration parameter ($q$), scalar expansion ($\theta$), shear scalar ($\sigma^2$) and the average anisotropy parameter ($A_m$) are defined as

$$q = -aa_{44}/a_4^2.$$  \hfill (13)

The sign of $q$ indicates whether the model inflates or not. A positive sign of $q$ corresponds to the standard decelerating model whereas the negative sign of $q$ indicates inflation. The recent observations of SN Ia [1,27] reveal that the present universe is accelerating and the value of deceleration parameter lies somewhere in the range $-1 < q < 0$.

$$\theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C},$$  \hfill (14)

$$\sigma^2 = \frac{3}{2}A_mH^2,$$  \hfill (15)

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i^2}{H} \right)^2,$$  \hfill (16)

where the mean Hubble parameter and $H_i (i = 1, 2, 3)$ represents the directional Hubble parameters in the directions of $x, y,$ and $z$ axes respectively.

3 Solution of Field Equations

Integrating equation (9), we get

$$C = lB,$$  \hfill (17)

where $l$ is constant of integration.

From equations (11), (8) and (9), we get

$$\delta = \eta.$$  \hfill (18)

With the help of equations (17) and (18), the set of equations reduces to

$$2B_{44}B + \frac{B_4^2}{B^2} + \frac{q^2}{A^2} - \frac{\omega}{2} \phi^2 \phi_4^2 = -w\rho,$$  \hfill (19)

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} - \frac{q^2}{A^2} - \frac{\omega}{2} \phi^2 \phi_4^2 = -(w + \delta)\rho,$$  \hfill (20)

$$2A_4B_4 + \frac{B_4^2}{B^2} - \frac{\omega}{2} \phi^2 \phi_4^2 = \rho,$$  \hfill (21)

$$\phi_{44} + \phi_4 \left( \frac{A_4}{A} + 2 \frac{B_4}{B} \right) + \frac{n}{2} \phi_4^2 = 0.$$  \hfill (22)
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There are four linearly independent equations (19)–(22) with six unknowns: $A$, $B$, $\phi$, $w$, $\rho$, $\delta$. In order to obtain exact solutions of the Einstein’s field equations, we normally assume a form for the matter content or suppose that the space-time admits killing vector symmetries. Solutions to the field equations may also be generated by applying a law of variation for Hubble’s parameter, which was first proposed by Berman [21] in FRW models and that yields a constant value of deceleration parameter (DP). Cunha and Lima [24] have favored recent acceleration and past deceleration with high degree of statistical confidence level by analyzing three SNe type-Ia samples. In order to match this observation, Singh and Debnath [25], Adhav et al. [26] have defined a special form of deceleration parameter for FRW metric as

$$q = -\frac{a_{44}a}{a_{4}^{2}} = -1 + \frac{\alpha}{1 + a^{\alpha}},$$  \hspace{1cm} (23)

where $\alpha > 0$, is a constant and $a$ is the mean scale factor of the universe. Thus, we have a model of universe which begins with a decelerating expansion and evolves into a late time accelerating universe which is in agreement with SNe Ia astronomical observations [28].

After solving (23) one can obtain the mean Hubble parameter $H$ as

$$H = \frac{a_{4}}{a} = k(1 + a^{-\alpha}),$$ \hspace{1cm} (24)

where $k$ is a constant of integration.

On integrating equation (24), we obtain the mean scale factor as

$$a = (e^{k\alpha t} - 1)\frac{1}{\alpha}. \hspace{1cm} (25)$$

We define the spatial volume $V$ of Bianchi type-VI$_{0}$ space-time as

$$V = a^{3} = AB^{2}. \hspace{1cm} (26)$$

Using equations (25) and (26), we have

$$(e^{k\alpha t} - 1)\frac{3}{\alpha} = AB^{2}. \hspace{1cm} (27)$$

In order to solve the above equations, we use a physical condition that the expansion scalar is proportional to shear scalar. We assume that the expansion $\theta$ in the model is proportional to the shears $\sigma$. This condition leads to

$$B = A^{n},$$ \hspace{1cm} (28)

where $n$ is arbitrary constant.

According to Thorne [29] observations of velocity redshift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic about
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30% range approximately (Kantowski and Sachs [30], Kristion and Sachs [31]) and redshift studies place the limit $\sigma/H \leq 0.30$. Collins [32] has discussed the physical significance of this condition for perfect fluid and barotropic equation of state in a more general case. Roy and Prakash [33], Roy and Banerjee [34], Bali and Singh [35] have proposed this condition in order to find exact solution of cosmological models.

Using equations (27) and (28), we obtain

\begin{equation}
A = (e^{\alpha kt} - 1) \frac{3}{\alpha(2n+1)},
\end{equation}

\begin{equation}
B = (e^{\alpha kt} - 1) \frac{3n}{\alpha(2n+1)}.
\end{equation}

Hence a Bianchi type-VI$_0$ space-time described by (1) can be written by using equations (29) and (30) in the form

\begin{equation}
\frac{\alpha}{2n+1} dx^2 + (e^{\alpha kt} - 1) \frac{6n}{\alpha(2n+1)} e^{2qz} dz^2.
\end{equation}

\section{Physical Parameters of Dark Energy Model}

It is well known that one can study the behavior of the physical and kinematical parameters either by observing the analytical expressions or by graphical representation.

Using equations (25) and (26), the spatial volume $V$ of the universe is given by

\begin{equation}
V = (e^{\alpha kt} - 1) \frac{3}{\alpha}.
\end{equation}

The spatial volume vanishes at $t = 0$. It expands exponentially as $t$ increase and becomes infinitely large as $t \rightarrow \infty$, as shown in Figure 1.

Using equations (29) and (30), the directional Hubble parameters are found as

\begin{equation}
H_x = (e^{\alpha kt} - 1) \frac{3k}{2n+1} e^{\alpha kt},
\end{equation}

\begin{equation}
H_y = H_z = (e^{\alpha kt} - 1) \frac{3nk}{2n+1} e^{\alpha kt}.
\end{equation}

Scalar field is given by

\begin{equation}
\phi = \left( \frac{(n+2)D}{2k(\alpha - 3)} \frac{(e^{\alpha kt} - 1)^{\frac{3}{n}}}{e^{\alpha kt}} \right)^{\frac{2}{n+2}}.
\end{equation}
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Using equations (21), (29) and (30), we obtain the energy density as

$$\rho = \frac{(18nk^2 + 9n^2k^2)}{(2n + 1)^2} \frac{e^{2\alpha kt}}{(e^{\alpha kt} - 1)^2}$$

$$- \frac{q^2}{(e^{\alpha kt} - 1)^{\frac{6}{\alpha}(2n+1)}} + \frac{\omega}{2} \frac{D^2}{(e^{\alpha kt} - 1)^{\frac{6}{\alpha}}}.$$  \hspace{1cm} (36)

Here we observe that the model starts with big bang having infinite density and as time increase (for finite time) the energy density $\rho$ tends to finite value. Hence after some finite time the model approaches to a steady state.

Using equations (19), (29) and (30), the EoS parameter is

$$w = \frac{-1}{\rho} \left\{ \frac{6\alpha nk^2}{(2n + 1)} \frac{e^{\alpha kt}}{(e^{\alpha kt} - 1)} \left[ \frac{e^{\alpha kt}}{(e^{\alpha kt} - 1)^{\frac{6}{\alpha}(2n+1)}} - 1 \right] + 1 \right\}$$

$$+ \frac{9n^2k^2}{(2n + 1)^2} \frac{e^{2\alpha kt}}{(e^{\alpha kt} - 1)^2} + \frac{q^2}{(e^{\alpha kt} - 1)^{\frac{6}{\alpha}(2n+1)}} - \frac{\omega}{2} \frac{D^2}{(e^{\alpha kt} - 1)^{\frac{6}{\alpha}}} \right\}. \hspace{1cm} (37)

Figure 3 depicts the variation of EoS parameter versus cosmic time for accelerating phase of Universe, as a representative case with appropriate choice of
Here we observe that the model starts with big bang having infinite density and as time increases (for finite time) the energy density $\rho$ tends to a finite value. Hence after some finite time the models approaches to steady state.

Using equations (19), (29) and (30), the EoS parameter as

$$0 < \omega < 1$$

$$\omega = \frac{\dot{w}}{\dot{\omega}}$$

Figure 2. Energy density vs time (for $\alpha = 0.5, 1, 2$).

Figure 3. EoS parameter vs time (for $\alpha = 0.5, 1$).
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constants of integration and other physical parameters. Equation (37) reveals that the EoS parameter is time dependent. The time dependent of the EoS parameter allows it to transit form $\omega > -1$ to $\omega < -1$ [36]. Here, since we assume that the equation of state is the function of time, the dark energy affects the CMB at early time. The study of the EoS parameter of the dark energy is very significant. We cannot determine the geometry of the universe without the knowledge of the EoS of dark energy. The SN Ia data suggests that $-1.67 < \omega < -0.62$ [37] while the limit imposed on $\omega$ by a combination of SN Ia data (with CMB anisotropy) and galaxy clustering statistics is $-1.33 < \omega < -0.79$ [38]. Figure 3 clearly shows that $\omega$ evolves within a range, which is in a good agreement with SN Ia and CMB observations. The plots of $\omega$ for $\alpha = 0.5$ and $\alpha = 1$ indicate that $\omega$ merges well with SN Ia and CMBR observations.

Using equations (20), (37), (29) and (30), the skewness parameter as

$$\delta = -1 \left\{ \frac{3 \alpha k^2}{(2n+1)} \frac{e^{\alpha kt}}{(e^{\alpha kt} - 1)} \left[ \frac{3}{e^{\alpha kt} - 1} \left( \frac{3}{\alpha(2n+1)} - 1 \right) + 1 \right] \right. $$

$$- \left. \frac{3n \alpha k^2}{(2n+1)} \frac{e^{\alpha kt}}{(e^{\alpha kt} - 1)} \left[ \frac{3n}{e^{\alpha kt} - 1} \left( \frac{3n}{\alpha(2n+1)} - 1 \right) + 1 \right] \right. $$

$$+ \left. \frac{9n^2 k^2}{(2n+1)^2} \frac{e^{2\alpha kt}}{(e^{\alpha kt} - 1)^2} - \frac{9nk^2}{(2n+1)^2} \frac{e^{2\alpha kt}}{(e^{\alpha kt} - 1)^2} \right\}. \quad (38)$$

The mean Hubble parameter $H$ is given by

$$H = \frac{ke^{\alpha kt}}{e^{\alpha kt} - 1}. \quad (39)$$

The expansion scalar is given by

$$\theta = \frac{3ke^{\alpha kt}}{e^{\alpha kt} - 1}. \quad (40)$$

It is observed that the expansion is infinite at $t = 0$, but as cosmic time $t$ increases, it decreases and halts at a finite value after some finite value of $t$.

The mean anisotropy parameter is given by

$$A_m = \frac{2(n-1)^2}{(2n+1)^2}. \quad (41)$$

Since the mean anisotropy parameter $A_m \neq 0$, the model does not approach isotropy for $n \neq 1$.  

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Figure 4. Expansion scalar vs. time (for $\alpha = 0.5, 1, 2$).

It is observed that the expansion is infinite at $t = 0$ but as cosmic time $t$ increases it decreases and halts at a finite value after some finite value of $t$.

The mean anisotropy parameter is given by

$$n_A = \frac{1}{2} \left( 1 + \frac{\alpha}{\sqrt{1 + \alpha^2}} \right)$$

Since the mean anisotropy parameter $0 \neq n_A$, the model does not approach isotropy for $1 \neq n$.

Figure 5. Shear scalar vs. time (for $\alpha = 0.5, 1$).

For this model, the shear scalar $\sigma \rightarrow 2\sigma$ as time $t \rightarrow 0$ and decreases to null as time increases.

The shear scalar is given by

$$\sigma = \frac{1}{2} \left( 1 + \frac{\alpha}{\sqrt{1 + \alpha^2}} \right)$$

(42)
The shear scalar is given by

$$\sigma^2 = \frac{3k^2(n - 1)^2}{(2n + 1)^2} \frac{e^{2\alpha kt}}{(e^{\alpha kt} - 1)^2}. \quad (42)$$

For this model, the shear scalar $\sigma^2 \to \infty$ as time $t \to 0$ and decreases to null as time increases.

Also, recent observations of SN Ia [1-2,27-28] reveal that the present universe is accelerating and the value of DP lies somewhere in the range $-1 < q < 0$. For $\alpha = 2$, the deceleration parameter $q$ is in the range $-1 \leq q \leq 0.5$ which matches with the observations made by [27,28] and the present day universe is undergoing accelerated expansion. It follows that in the derived model, one can choose the values of DP consistent with the observations.

5 Conclusions

Dark energy cosmological models are, recently, playing a vital role in the discussion of accelerated expansion of the universe in general relativity. With the advent of alternative theories of gravitation study of these models is gaining importance. In this paper, we have investigated Bianchi type-$VI_0$ cosmological
model in Saez-Ballester [16] scalar-tensor theory of gravitation in the presence of anisotropic dark energy and special form of deceleration parameter. It is observed that the EoS parameter $\omega$ is time dependent, it can be a function of redshift $z$ or scale factor $V$. If the present work is compared with the experimental results already mentioned in the introduction, one can conclude that the limit of $\omega$ provided by (37) may be accommodated with the acceptable range of EoS parameter. The model obtained and presented here represents an accelerating and expanding cosmological model of the universe. Also this model is consistent with the recent observations of type-Ia supernovae.

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References

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