Heating of the Solar Corona by Alfvén Waves – Self-Induced Opacity

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Received 26 March 2013

Abstract. Static distributions of temperature and wind velocity at the transition region are calculated within the framework of magnetohydrodynamics (MHD) of completely ionized hydrogen plasma. The numerical solution of the derived equations gives the width of the transition layer between the chromosphere and the corona as a self-induced opacity of high-frequency Alfvén waves (AW). The domain wall is direct consequence of the self-consistent MHD treatment of AW propagation. The low-frequency MHD waves coming from the Sun are strongly reflected by the narrow transition layer, while the high-frequency waves are absorbed - that is why we predict considerable spectral density of the AWs in the photosphere. The idea that Alfvén waves might heat the solar corona belongs to Alfvén, we simply solved the corresponding MHD equations. The comparison of the solution to the experiment is crucial for revealing the heating mechanism.

PACS codes: 52.35.-g, 95.30.Qd

1 Alfvén Model for Corona Heating

The discovery of the lines of the multiply ionized iron in the solar corona spectrum [1] posed an important problem for the fundamental physics – what is the mechanism of the heating of the solar corona and why the temperature of the corona is 100 times larger than the temperature of the photosphere.

The first idea by Alfvén [2] was that Alfvén waves (AW) [3] are the mechanism for heating the corona. AW are generated by the turbulence in the convection zone and propagate along the magnetic field lines. Absorption is proportional to $\omega^2$ and the heating comes from high-frequency AW. The low frequency AW reach the Earth orbit and thanks to the magnetometers on the various satellites we “hear” the basses of the great symphony of solar turbulence.
The purpose of the present work is to examine whether the initial idea is correct and to solve the MHD equations which give the dependence of the temperature on the height $T(x)$ and the related velocity of the solar wind $U(x)$.

Our starting point are the MHD equations for the velocity field $v$ and magnetic field $B$

\[ \partial_t \rho + \text{div} \, j = 0, \quad j = \rho v, \]  
\[ \partial_t \left( \frac{1}{2} \rho v^2 + \epsilon + \frac{B^2}{2 \mu_0} \right) + \text{div} \, q = 0, \]  
\[ \partial_t (\rho v) + \nabla \cdot \Pi = 0, \]

where

\[ q = \rho \left( \frac{1}{2} v^2 + h \right) v + v \cdot \Pi^{(\text{visc})} - \chi \nabla T + S \]

is the energy density flux, $\rho$ is the mass density, $\epsilon$ is the internal energy density, $\chi$ is the thermal conductivity, $h$ is the enthalpy per unit mass;

\[ S = \frac{1}{\mu_0} \left[ B \times (v \times B) - \nu_m B \times (\nabla \times B) \right], \]

is the Pointing vector and $\nu_m \equiv c^2 \varepsilon_0 \varrho$ is the magnetic diffusion determined by Ohmic resistance $\varrho$ and vacuum susceptibility $\varepsilon_0$; vacuum permeability is $\mu_0$. For hot enough plasma $\varrho$ is negligible and we ignore it hereafter. The total momentum flux

\[ \Pi = \rho v v + P \mathbb{1} + \Pi^{(\text{visc})} + \Pi^{(\text{Maxw})} \]

is a sum of the inviscid part $\rho v v + P$ of the fluid, with pressure $P$,

\[ \Pi^{(\text{visc})}_{ik} = -\eta \left( \partial_i v_k + \partial_k v_i - \frac{2}{3} \delta_{ik} \nabla \cdot v \right) - \zeta \delta_{ik} \nabla \cdot v, \]

the viscous part of the stress tensor, with viscosity $\eta$ and second viscosity $\zeta$, and lastly, the Maxwell stress tensor

\[ -\Pi^{(\text{Maxw})}_{ik} = \frac{1}{\mu_0} \left( B_i B_k - \frac{1}{2} B^2 \delta_{ik} \right), \]

with $\delta_{ik}$ the Kronecker delta. We model coronal plasma with completely ionized hydrogen plasma

\[ \chi = 0.9 \frac{T^{5/2}}{e^4 m_e^{1/2} \Lambda}, \quad \eta = 0.4 \frac{m_p^{1/2} T^{5/2}}{e^4 \Lambda}, \quad \zeta \approx 0, \]

\[ \Lambda = \ln \left( \frac{r_D T}{e^2} \right), \quad \frac{1}{r_D^2} = \frac{4 \pi e^2 n_{tot}}{T}, \quad e^2 \equiv \frac{q_e^2}{4 \pi \varepsilon_0} \]

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where $q_e$ is the electron charge, $m_e$ is the mass of electron, $m_p$ is the proton mass, $T$ is the temperature and $n_{\text{tot}} = n_e + n_p$ is the total density of electrons and protons; $\rho = m_p n_p$. We suppose that $\mu_0 = 4\pi$ and $\varepsilon_0 = 1/4\pi$, but in the practical system all formulae are the same; as well as in Heaviside-Lorentz units, where $\mu_0 = 1$ and $\varepsilon_0 = 1$. As we mentioned above

$$\nu_m = \frac{e^2}{4\pi} \frac{e^2 m_e^{1/2} \Lambda}{0.6 T^{3/2}} \ll \nu_k \equiv \frac{\eta_0}{\rho} = \frac{0.4 T^{5/2}}{e^4 m_p^{1/2} n_p \Lambda}; \quad (10)$$

i.e. the hot hydrogen plasma is sticky, dilute, and “superconducting” $\nu_m \approx 0$. Let us mention also the relations $\kappa_0 = 1.5 T/q_e^2$ and $\eta/\kappa \approx \frac{4}{3} \sqrt{m_e m_p}$, $\rho = \frac{1}{4\pi \varepsilon_0} \frac{e^2 m_e^{1/2} \mathcal{L}}{0.6 T^{3/2}}$. \quad (11)

## 2 MHD Equations and Energy Fluxes

The time derivative $\partial_t B$ which implicitly participate in the energy conservation Eq. (2) at zero Ohmic resistivity obeys the equation

$$d_t B = B \cdot \nabla v - B \text{ div } v, \quad d_t \equiv \partial_t + v \cdot \nabla. \quad (12)$$

Analogously the momentum equation Eq. (3) can be rewritten by the substantial derivative

\[ \rho d_t v_i = -\partial_i P + \partial_k \left\{ \eta \left( \partial_k v_i + \partial_i v_k - \frac{2}{3} \delta_{ik} \partial_j v_j \right) \right\} \]

\[ + \partial_i (\zeta \partial_j v_j) - \frac{1}{\mu_0} (B \times \text{rot } B)_i. \quad (13) \]

In our model we consider AW propagating along magnetic field lines $B_0$. We focus our attention on the narrow transition layer, where the static magnetic field is almost homogeneous and the waves are within acceptable accuracy one-dimensional. For the velocity and magnetic fields we assume

$$v(t, x) = U(x) \hat{x} + u(t, x) \hat{z},$$

$$B(t, x) = B_0 \hat{x} + b(t, x) \hat{z}, \quad (14)$$

with homogeneous magnetic field $B_0 \hat{x}$ perpendicular to the surface of the Sun. The transverse wave amplitudes of the velocity $u(t, x)$ and magnetic field $b(t, x)$ we represent with the Fourier integrals

$$u(t, x) = \int_{-\infty}^{\infty} \hat{u}(\omega, x) e^{-i\omega t} \frac{d\omega}{2\pi}, \quad (15)$$

$$b(t, x) = \int_{-\infty}^{\infty} \hat{b}(\omega, x) e^{-i\omega t} \frac{d\omega}{2\pi}. \quad (16)$$
For illustrative purposes it is convenient to consider monochromatic AW with $u(t, x) = \hat{u}(x)e^{-i\omega t}$ and $b(t, x) = \hat{b}(x)e^{-i\omega t}$.

### 2.1 Wave equations

For linearized waves the general MHD equations Eq. (13) and Eq. (12) give the following system for $\hat{u}(x)$ and $\hat{b}(x) = \frac{\hat{b}(x)}{B_0}$:

\begin{align}
(-i\omega + U d_x) \hat{u} &= V_A^2 d_x \hat{b} + \frac{1}{\rho} d_x (\eta d_x \hat{u}) , \quad (17) \\
-\omega \hat{b} &= d_x \hat{u} - d_x (U \hat{b}), \quad (18)
\end{align}

where

\begin{equation}
V_A(x) = B_0/\sqrt{\mu_0 \rho(x)} \quad (19)
\end{equation}

is the Alfvén velocity. In our numerical analysis we solve the first order linear system of equations

\begin{equation}
-i d_x |\Psi\rangle = \frac{i}{\nu k U} M |\Psi\rangle = K |\Psi\rangle , \quad (20)
\end{equation}

|\Psi\rangle \equiv \begin{pmatrix} \hat{u} \\ \hat{b} \\ \hat{w} \end{pmatrix} , \quad K = \frac{i}{\nu k U} M ,

\langle \Psi | \equiv |\Psi\rangle^\dagger = (\hat{u}^*, \hat{b}^*, \hat{w}^*) ,

where $\hat{w} \equiv d_x \hat{u}$, and

\begin{equation}
M \equiv \begin{pmatrix}
0 & 0 & -\nu_k U \\
0 & \nu_k (-i\omega + d_x U) & -\nu_k \\
i\omega - V_A^2 (-i\omega + d_x U) & (V_A^2 - U^2) + \frac{U}{\rho} d_x \eta
\end{pmatrix} . \quad (21)
\end{equation}

For homogeneous medium with constant $\eta$, $\rho$, $V_A$, and $U$, in short for constant wave-vector matrix $K$, the exponential substitution $\Psi \propto \exp(i k x)$ in Eq. (20) or equivalently Eq. (17) and Eq. (18) gives the secular equation

\begin{equation}
i\nu_k U \det (K - k I) = \omega_D (\omega_D + i\nu_k k^2) - V_A^2 k^2 = 0 , \quad (22)
\end{equation}

where $\omega_D \equiv \omega - kU$ is the Doppler shifted frequency. This secular equation gives the well-known dispersion $\omega_D (\omega_D + i\nu_k k^2) = V_A^2 k^2$ of the AW. This equation is quadratic with respect to $\omega$ and cubic with respect to $k$.

### 2.2 Wind variables

We solve the wave equation Eq. (20) from “Sun’ surface” $x = 0$ to some distance large enough $x = l$, where the short wavelength AW are almost absorbed.
This distance is much bigger than the \textit{width of the transition layer} $\lambda$, but much smaller than solar radius. The considered one-dimensional $0 < x < l$ time-independent problem has three integrals corresponding to the three conservation laws related to mass, energy and momentum. The mass conservation law Eq. (1) gives the constant flow

$$j = \rho(x)U(x) = \rho_0 U_0 = \rho_l U_l = \text{const},$$

(23)

where $\rho_0 = \rho(0)$, $\rho_l = \rho(l)$, $U_0 = U(0)$, and $U_l = U(l)$. The energy conservation law reduces to a constant flux along the $x$-axis

$$q = \tilde{q} = q_{\text{ideal}}(x) + \tilde{q}(x) = \rho U \left( \frac{1}{2} U^2 + h \right) + \tilde{q} = \text{const.}$$

(24)

Here the first term describes the energy of the ideal wind, i.e. a wind from an ideal (inviscid) fluid. The second term $\tilde{q}(x)$ includes all other energy fluxes; in our notations tilde will denote sum of the non-ideal (dissipative) terms of the wind and wave terms. In detail the non-ideal part of the energy flux $\tilde{q}(x)$ consists of: the wave kinetic energy $\propto |\hat{u}|^2$, viscosity (wind $\propto \frac{4}{3} \eta + \zeta$ and wave $\propto \eta$ components), heat conductivity $\propto \kappa$, and Pointing vector $\propto \hat{b}^*$. \begin{equation}
\tilde{q}(x) = \frac{j}{4} |\hat{u}|^2 - \xi U d_x U - \frac{1}{4} \eta d_x |\hat{u}|^2 - \kappa d_x T + \frac{1}{2 \mu_0} (U |\hat{b}|^2 - B_0 \Re(\hat{b}^* \hat{u})),$$
\end{equation}

(25)

where $\xi = \frac{4}{3} \eta + \zeta$. Here time averaged energy flux is represented by the amplitudes of the monochromatic oscillations, this is a standard procedure for alternating current processes. In our case we have, for example, $\langle |\hat{u}|^2 \rangle_t = \langle |\Re(\hat{u})|^2 \rangle_t = \langle |\frac{1}{2} (\hat{u} + \hat{u}^*)|^2 \rangle_t = \frac{1}{2} |\hat{u}|^2$. The other terms from Eq. (4) are averaged in a similar way in the equation above.

The momentum conservation law Eq. (6) gives constant flux $\Pi = \Pi_{xx}$

$$\Pi = \Pi_{\text{wind}}(x) + \tilde{\Pi}(x) = \rho U U + P + \tilde{\Pi},$$

(26)

the sum of the ideal wind fluid and the other terms

$$\tilde{\Pi}(x) = \frac{1}{4 \mu_0} \left| \hat{b} \right|^2 - \xi d_x U,$$

(27)

which take into account the wave part of the Maxwell stress tensor $\propto b^2$ and viscosity of the wind $\propto \xi$.

We have to solve the hydrodynamic problem for calculation of wind velocity and temperature at known energy and momentum fluxes. The problem is formally reduced to analogous one for a jet engine, cf. Ref. [4]. We approximate the corona as completely ionized hydrogen plasma, i.e. electrically neutral mixture of electrons and protons. The experimental data tell that proton temperature $T_p$...
is higher than the electron one \( T_e \). This is an important hint that heating goes through the viscosity determined mainly by protons. However for illustration purpose and simplicity we assume proton and electron temperatures to be equal \( T_e = T_p = T \). For such an ideal (in thermodynamic sense) gas the local sound velocity is

\[
\frac{c_s^2(x)}{c_s^2(0)} = \frac{c_p P}{\gamma \langle m \rangle}, \quad \gamma = \frac{c_p}{c_s} = \frac{5}{3},
\]

\[
\langle m \rangle = \frac{n_p m_p + n_e m_e}{n_p + n_e} \approx \frac{1}{2} m_p, \quad n_e = n_p = \frac{1}{2} n_{\text{tot}},
\]

\[
P = n_{\text{tot}} T = \rho T \langle m \rangle = j \frac{U}{\langle m \rangle},
\]

\[
h = c_p T \frac{\gamma T}{\langle m \rangle} = \frac{\varepsilon + P}{\rho},
\]

where, as we mentioned earlier, \( h \) is the enthalpy per unit mass and \( \varepsilon \) is the density of internal energy.

In order to alleviate the final formulae we introduce two dimensionless variables \( \chi \) and \( \tau \) which represent the non-ideal part of the energy and momentum flux respectively

\[
\chi(x) \equiv \frac{\tilde{q}(x)}{\rho_0 U_0^2} |_x, \quad \tau(x) \equiv \frac{\tilde{\Pi}(x)}{\rho_0 U_0^2} |_x,
\]

and analogously for the wind velocity and temperature

\[
\bar{U}(x) \equiv \frac{U(x)}{U_0}, \quad \Theta(x) \equiv \frac{T(x)}{\langle m \rangle U_0^2}, \quad \Theta_0 = \Theta(0),
\]

where \( U_0 = U(0) \). The energy and momentum constant fluxes Eq. (24) and Eq. (26) in the new notation take the form

\[
\frac{q - \tilde{q}(0)}{\rho_0 U_0^2} = \frac{1}{2} U^2 + c_p \Theta - \chi = \frac{1}{2} + c_p \Theta_0,
\]

\[
\frac{\Pi - \tilde{\Pi}(0)}{\rho_0 U_0^2} = U + \Theta / \bar{U} - \tau = 1 + \Theta_0.
\]

From the second equation we express the dimensionless temperature \( \Theta \) and substitute in the first one. Solving the quadratic equation for the wind velocity \( U \), we derive

\[
\bar{U}(x) = \frac{1}{\gamma + 1} \left( \gamma + s^2 + \tau(x) - \sqrt{\mathcal{D}(x)} \right),
\]

where for the discriminant we have

\[
\mathcal{D} = (s^2 - 1)^2 - 2(\gamma^2 - 1)\gamma \tau \left[ \gamma \tau + 2(\gamma + s^2) \right],
\]

\[
s^2 \equiv \frac{c_s^2(0)}{U_0^2} = \gamma \Theta_0, \quad c_s^2(x) = \left( \frac{\partial P}{\partial \rho} \right)_{\gamma} = \gamma T(x) \frac{\gamma T}{\langle m \rangle}.
\]
Here $\gamma$ is the constant ratio of the heat capacities, and $s \equiv c_s(0)/U_0$ is the ratio of the sound and wind velocity at $x = 0$. We suppose that initial wind velocity is very small $U(0) \ll c_s(0)$. The velocity distribution Eq. (33) can be substituted in Eq. (32) and we derive the dimensionless equation for the temperature distribution

$$\Theta(x) = \Theta(x) \left( 1 + \Theta_0 + \gamma \tau(x) - U(x) \right), \quad \Theta_0 = \frac{s^2}{\gamma}.$$  

(35)

The solutions for velocity $U(x)$ Eq. (33) and temperature $T(x)$ Eq. (35) distributions are important ingredients in our analysis and derivation of the self-consistent picture of the solar wind.

### 2.3 Boundary conditions for the waves

At known background wind variables $U(x)$ and $T(x)$ we can solve the wave equation Eq. (20) for run-away AW at $x = l$. As we will see later the run-away boundary condition Eq. (50) corresponds to right propagating AW at the right boundary of the interval. The wave equation Eq. (20) is extremely stiff at small viscosity, and numerical solution is possible to be obtained only downstream from $x = 0$ to $x = l$. We have to find the linear combination of left and right propagating waves at $x = 0$, which gives the run-away condition at $x = l$.

The solution of wave equation according to Eq. (24) determines the energy flux related to the propagation of AW

$$\tilde{q}_{\text{wave}}(\Psi(x)) \equiv \langle \Psi | g | \Psi \rangle = \frac{j}{4} |\tilde{u}|^2 - \frac{1}{2} \eta \Re(\tilde{u}^* \tilde{w}) + \frac{B_0^2}{2\mu_0} (U |\tilde{b}|^2 - \Re(\tilde{b}^* \tilde{u})).$$  

(36)

where

$$g(x) = \begin{pmatrix}
\frac{1}{4}j & -\frac{B_0^2}{4\mu_0} & -\frac{1}{4} \eta \\
-\frac{B_0^2}{4\mu_0} & \frac{U B_0^2}{2\mu_0} & 0 \\
-\frac{1}{4} \eta & 0 & 0
\end{pmatrix}. $$  

(37)

Here $j$-term represents the kinetic energy of the wave, $\eta$-terms come from the viscous part of the wave energy flux, and $B_0$-terms describe the Pointing vector of the wave.

In order to take into account the boundary condition at $x = l$ we calculate the eigenvectors of the matrix $K$, which according to Eq. (20) determine the wave propagation in a homogeneous fluid with amplitude $\propto \exp(i k x)$. Then the eigenvalues of $K$ give the complex wave-vectors

$$k = k' + ik'' = \text{eigenvalue}(K),$$  

(38)
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i.e. \[ \det (K - kI) = 0. \] (39)

The three eigenvectors \(|L\rangle, |D\rangle\) and \(|R\rangle\) are ordered by spatial decrements of their eigenvalues
\[ k''_L < k''_R < k''_D, \] (40)

and are normalized by the conditions
\[ -\langle L|g|L \rangle = \langle R|g|R \rangle = \langle D|g|D \rangle = 1, \] (41)

where the sign corresponds to the direction of wave propagation. Notation \(L\) corresponds to the left propagating wave, \(R\) – to the right propagating wave, and \(D\) for a overdamped at small viscosity mode.

For technical purposes we introduce the matrix notations
\[ |L\rangle = \begin{pmatrix} L_u(x) \\ L_b(x) \\ L_w(x) \end{pmatrix}, \quad |R\rangle = \begin{pmatrix} R_u(x) \\ R_b(x) \\ R_w(x) \end{pmatrix}, \quad |D\rangle = \begin{pmatrix} D_u(x) \\ D_b(x) \\ D_w(x) \end{pmatrix}. \] (42)

For low enough frequencies \(\omega \to 0\) and wind velocities the modes describe: 1) right-propagating AW with \(k''_R \approx \omega/V_A\) and small \(k''_R \approx \nu_k \omega^2/2V_A^3 \ll k''_R\), 2) left propagating wave \(k_L = -k_R\), and a diffusion overdamped mode \(k''_D \approx V_A^2/\nu_k U \gg k''_R\) which describes the drag of a static perturbation by the slow wind \(U \ll V_A\) in a fluid with small viscosity. In this low frequency and long wavelength limit the stiffness ratio of the eigenvalues is very large
\[ r_{DR} = \frac{|k_D|}{|k_R|} \approx \frac{k''_D}{k''_R} \approx \frac{V_A^3}{\nu_k U \omega} \gg 1. \] (43)

The strong inequality is applicable to the chromosphere, where the viscosity of the cold plasma is very low. As we emphasized the wave equation Eq. (20) is a very stiff system and indispensably has to be solved downstream from the chromosphere \(x = 0\) to the corona \(x = l\) using algorithms for stiff systems. Let
\[ |\psi_L(x)\rangle = \begin{pmatrix} u_L(x) \\ b_L(x) \\ w_L(x) \end{pmatrix}, \quad |\psi_R(x)\rangle = \begin{pmatrix} u_R(x) \\ b_R(x) \\ w_R(x) \end{pmatrix} \] (44)

are the solutions of the wave equation Eq. (20) with boundary conditions
\[ |\psi_L(0)\rangle = |L(0)\rangle, \quad |\psi_R(0)\rangle = |R(0)\rangle. \] (45)

We look for a solution as linear combination
\[ \psi(x) = \psi_R(x) + r \psi_L(x), \] (46)
in other words we suppose that from the low viscosity chromosphere plasma do not come overdamped diffusion modes. The strong decay rate makes them negligible at $x = 0$. Physically this means that AW (R-modes) are coming from Sun and some of them are reflected from the transition layer (L-modes)

$$\psi(0) = |R(0)| + r|L(0)|.$$  

(47)

Analogously for the configuration of open corona we have to take into account the run-away boundary condition for which we suppose zero amplitude for the wave coming from infinity

$$\psi(l) = \tilde{t}|R(l)| + \tilde{c}|D(l)|.$$  

(48)

Written by components

$$\begin{pmatrix} u_R(l) \\ b_R(l) \\ w_R(l) \end{pmatrix} + r \begin{pmatrix} u_L(l) \\ b_L(l) \\ w_L(l) \end{pmatrix} = \tilde{t} \begin{pmatrix} R_R(l) \\ R_L(l) \\ R_w(l) \end{pmatrix} + \tilde{c} \begin{pmatrix} D_R(l) \\ D_L(l) \\ D_w(l) \end{pmatrix},$$  

(49)

this boundary condition gives a linear system of equation for the reflection coefficient $r$, transmission coefficient $\tilde{t}$ and the mode-conversion coefficient $\tilde{c}$. The solution of Eq. (20) is depicted in Figure 2.

For $l \to \infty$ when $\exp[-k_R^2(l)/l] \ll 1 \exp[-k_L^2(l)/l]$ the amplitude of D-mode is negligible and the run-away boundary condition reads

$$\psi(l) = \psi_R(l) + r\psi_L(l) \approx \tilde{t}|R(l)|,$$  

(50)

Figure 1: Solutions of the wave velocity for $\omega = \{700, 849, 1029\}$ Hz.
Figure 2: Solutions of the wave equations Eq. (20) with boundary conditions Eq. (47) and Eq. (48) at different frequencies: $\omega = \{700, 849, 1029, 1247, 1512, 1833, 2222, 2693\}$ Hz. The darkest lines correspond to the smallest $\omega$, the lightest correspond to the largest $\omega$. 
or by components
\[
\begin{pmatrix}
  u_R(l) \\
  b_R(l)
\end{pmatrix}
+ r \begin{pmatrix}
  u_L(l) \\
  b_L(l)
\end{pmatrix}
= \tilde{t} \begin{pmatrix}
  R_u(l) \\
  R_b(l)
\end{pmatrix}.
\] (51)

These systems give the amplitudes of the reflected wave \(r\) and the transmitted wave \(\tilde{t}\) in the solution Eq. (46). For this solution we have the energy fluxes
\[
T \equiv \langle \psi(l)|g(l)|\psi(l) \rangle = |\tilde{t}|^2 + |\tilde{c}|^2 + (\tilde{c}^*\langle D(l)|g(l)|R(l) \rangle + c.c.),
\]
(52) and
\[
1 - R \equiv \langle \psi(0)|g(0)|\psi(0) \rangle = 1 - |r|^2 + (r^*\langle L(0)|g(0)|R(0) \rangle + c.c.).
\] (53)

Then we introduce the absorption coefficient
\[
A \equiv - \langle \psi(x)|g(x)|\psi(x) \rangle |l_0^{-1} = 1 - R - T.
\] (54)

The described solution is normalized by unite energy flux of the R-wave. If we wish to fix energy flux of the right propagating wave to be \(q_{\text{wave}}(0)\), we have to make the renormalization
\[
\Psi(x) = A_{\text{wave}} \psi(x).
\] (55)

Now using \(\Psi(x)\) we can calculate the wave part of the energy flux Eq. (36) and the wave part of the momentum flux
\[
\tilde{\Pi}_{\text{wave}}(x) \equiv \frac{1}{4\mu_0} \left| \hat{b}(x) \right|^2.
\] (56)

This section is written in dimensional variables, but all equations can be easily converted in dimensionless variables as is done in the next sub-sub-section.

**Dimensionless wave variables**

Using mechanical units for length \(l\), velocity \(U_0\) and density \(\rho_0\) we can convert all equations in dimensionless form. The formulae remain almost the same and we wish to mention only the differences. Introducing dimensionless density
\[
\rho(x) = \rho(x)/\rho_0 = 1/U(x)
\] (57) and wave energy flux
\[
Q_{\text{wave}}(0) = \frac{q_{\text{wave}}(0)}{\rho_0^3 U_0^3} = (1 - R) |A_{\text{wave}}|^2,
\] (58)

we have dimensionless matrices
\[
\mathbf{M} = \begin{pmatrix}
  0 & 0 & -\rho U \\
  0 & \rho(-i\omega + \mathbf{W}) & -\rho \\
  i\omega U - \nabla^2 A(-i\omega + \mathbf{W}) & \left(\nabla^2 - U^2\right) + U^2 d\mathbf{v} \end{pmatrix}.
\] (59)
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and

$$\bar{g} \equiv \frac{1}{4} \left( \begin{array}{ccc} \frac{1}{2} & -a^2 & -\bar{\eta}(x) \\ -a^2 & 2a^2\bar{U}(x) & 0 \\ -\bar{\eta}(x) & 0 & 0 \end{array} \right),$$

$$V_A^2(x) = a^2\bar{U}(x), \quad V_A^2(0) = a^2\bar{U}(0) = a^2.$$ \hspace{1cm} (60)

For the dimensionless energy flux we have

$$Q_{\text{wave}}(x) = \frac{1}{4}|\hat{\mathbf{u}}|^2 + \frac{a^2}{2}(\bar{U}|\hat{\mathbf{b}}|^2 - \text{Re}(\hat{\mathbf{b}}^* \hat{\mathbf{w}})) - \frac{1}{2}\bar{\eta}\text{Re}(\hat{\mathbf{w}}^* \hat{\mathbf{w}}) = \langle \bar{\Psi} | \bar{g} | \bar{\Psi} \rangle,$$ \hspace{1cm} (62)

where

$$\hat{\mathbf{u}} = \frac{\hat{u}}{U_0}, \quad \hat{\mathbf{w}} = \frac{\hat{u} \times \hat{b}}{U_0}, \quad \hat{\mathbf{w}} = \frac{\omega}{U_0}.$$ \hspace{1cm} (63)

Then for dimensional wave energy flux we have

$$q_{\text{wave}}(x) = \rho_0U_0^3 Q_{\text{wave}}(x),$$ \hspace{1cm} (64)

and analogously for the momentum flux of the wave

$$P_{\text{wave}}(x) = \rho_0U_0^2 P_{\text{wave}}(x), \quad P_{\text{wave}} = \frac{1}{4}|\hat{\mathbf{b}}|^2.$$ \hspace{1cm} (65)

in the next section we will consider all parts of the energy and momentum fluxes.

2.4 Total wave fluxes

The total energy and momentum fluxes are integrals over all frequencies. From Eq. (62) we have

$$Q_{\text{wave}}(x) = \int_0^{\infty} W_\omega(\langle \psi_\omega | \bar{g} | \psi_\omega \rangle) d\omega,$$ \hspace{1cm} (66)

where $W_\omega$ is the spectral density of the waves. We can construct the total energy flux of the waves as

$$Q_{\text{wave}}(x) = \int_0^{\infty} W_\omega(\langle \psi_\omega | \bar{g} | \psi_\omega \rangle) \frac{d\omega}{2\pi},$$ \hspace{1cm} (67)

$$Q_{\text{wave}}(0) \equiv Q_0 = \sum_{\omega>0} W_\omega(\langle \psi(0) | \bar{g}(0) | \psi(0) \rangle).$$ \hspace{1cm} (68)

In our model we have chosen spectral density in the form $W_\omega = C/\omega^2$. $C$ is the unknown parameter of the theory, which we vary until we reproduce the right increase of the temperature in the transition layer. Note that here we have used...
the dimensional frequency and $C$ is constant for all frequencies. If we want to use the dimensionless one, then the parameter $C$, that has a dimension of $\omega^2$, has to become dimensionless, and then dependent of $\varkappa$. If we know the initial total energy flux of the waves, we can calculate the spectral density as

$$W_\varkappa = Q_0 \sum_{\varkappa > 0} \langle \psi_\varkappa(0) | \varpi(0) | \psi_\varkappa(0) \rangle.$$  \hfill (69)

Analogously to Eq. (67), the total momentum flux is calculated from Eq. (65) as

$$P_{\text{total}}(x) = \sum_{\varkappa > 0} W_\varkappa \bar{P}_{\text{wave}, \varkappa}(x) = \int_{\varkappa=0}^{\infty} W_\varkappa \bar{P}_{\text{wave}, \varkappa}(x) \frac{d\varkappa}{2\pi}. \hfill (70)$$

In Figure 3 are shown wave fluxes of single waves for several frequencies corresponding to the solutions in Figure 1, while in Figure 5 are the total energy and momentum fluxes calculated according to Eq. (67) and Eq. (70).

(a) Energy fluxes (in J/m$^2$.s) of the waves according to Eq. (66) and wave solutions from Figure 2.

(b) Frequency dependent momentum fluxes (in kg/m.s$^2$) according to Eq. (65), Eq. (81) and Eq. (85) (multiplied by $C^2/\omega$ as in Eq. (70)) and wave solutions from Figure 2.

Figure 3: Energy and momentum wave fluxes for different frequencies (see Figure 2). The darkest lines correspond to the smallest $\omega$, the lightest correspond to the largest $\omega$. 

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2.5 Mass, energy and momentum fluxes

In the one-dimensional model which we analyze the conservation laws Eq. (1), Eq. (2) and Eq. (3) are converted in three integrals of our dynamic system describing the mass \( j = \rho_0 U_0 \tilde{j} \), energy \( q = \rho_0 U_0^3 Q \), and momentum \( \Pi = \rho_0 U_0^3 \tilde{P} \) fluxes

\[
\tilde{j} = \tilde{U} \tilde{p} = 1, \\
Q = \frac{1}{2} \tilde{T}^2 + c_p \Theta_0 T - \Xi \Theta_0 d \tilde{T} - \left( \frac{4}{3} \eta + \zeta \right) \tilde{U} d \tilde{T} \\
+ \sum_{\omega > 0} \mathcal{W}_{\omega} \left( \frac{1}{4} \left| \tilde{\pi}_{\omega} \right|^2 + \frac{a^2}{2} \left( \tilde{U} \left| \tilde{b}_{\omega} \right|^2 - \text{Re} \left( \tilde{b}_{\omega}^{*} \tilde{b}_{\omega} \right) \right) \right) \\
- \sum_{\omega > 0} \mathcal{W}_{\omega} \frac{1}{2} \eta \text{Re} \left( \tilde{\pi}_{\omega}^{*} \tilde{\pi}_{\omega} \right) = \text{const}, \\
\tilde{P} = \tilde{U} + \frac{\Theta_0 T}{\tilde{U}} - \left( \frac{4}{3} \eta + \zeta \right) d \tilde{T} + \sum_{\omega > 0} \mathcal{W}_{\omega} \frac{1}{4} \left| \tilde{b}_{\omega} \right|^2 = \text{const}. 
\]

Here we can recognize the energy flux of an ideal inviscid gas (Figure 4a)

\[
Q_{\text{ideal wind}} = \frac{1}{2} \tilde{T}^2 + c_p \Theta_0 T, \quad \Theta = \Theta_0 T, 
\]

dissipative energy flux (Figure 4b) of the wind related to heat conductivity and viscosity

\[
Q_{\text{diss wind}} = -\Xi \Theta_0 d \tilde{T} - \left( \frac{4}{3} \eta + \zeta \right) \tilde{U} d \tilde{T}, 
\]

the non-absorptive part of wave energy flux

\[
Q_{\text{ideal wave}} = \sum_{\omega > 0} \mathcal{W}_{\omega} \left( \frac{1}{4} \left| \tilde{\pi}_{\omega} \right|^2 + \frac{a^2}{2} \left( \tilde{U} \left| \tilde{b}_{\omega} \right|^2 - \text{Re} \left( \tilde{b}_{\omega}^{*} \tilde{b}_{\omega} \right) \right) \right), 
\]

and the absorptive part of the wave energy flux

\[
Q_{\text{diss wave}} = -\sum_{\omega > 0} \mathcal{W}_{\omega} \frac{1}{2} \eta \text{Re} \left( \tilde{\pi}_{\omega}^{*} \tilde{\pi}_{\omega} \right) 
\]

proportional to viscosity. Analogously for the momentum flux we have:

\[
P_{\text{ideal wind}} = \tilde{U} + \frac{\Theta_0 T}{\tilde{U}}, \\
P_{\text{diss wind}} = -\left( \frac{4}{3} \eta + \zeta \right) d \tilde{T}, \\
P_{\text{ideal wave}} = \sum_{\omega > 0} \mathcal{W}_{\omega} \frac{1}{4} \left| \tilde{b}_{\omega} \right|^2, \\
P_{\text{diss wave}} = 0
\]
(a) Energy flux of the ideal wind according to Eq. (74).

(b) Energy flux of the dissipative wind according to Eq. (75).

(c) Total energy flux (Eq. (72)) normed to 1.

Figure 4: Energy fluxes in J/m$^2$.s.
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(a) Total energy flux (in J/m².s) of the waves calculated according to Eq. (67).

(b) Total momentum flux (in kg/m.s²) of the waves calculated according to Eq. (70).

Figure 5: Total wave energy and momentum fluxes.

for the transversal AW. As a rule the dissipative fluxes are against the non-dissipative ones. One can introduce non-ideal wind energy flux

\[ \hat{Q} \equiv Q_{\text{wind}}^{\text{nonideal}} = Q_{\text{wind}}^{\text{diss}} + Q_{\text{wave}}^{\text{total}} = Q - Q_{\text{wind}}^{\text{ideal}} = -R dT + \left( \frac{4}{3} \eta + \zeta \right) \hat{U} dT \hat{U} - \sum_{\omega > 0} W_\omega \left( \frac{1}{2} \Re \left( \hat{w}_\omega \hat{v}_\omega \right) + \frac{a^2}{2} \left( \hat{U} |\hat{\psi}_\omega| - \Re \left( \hat{b}_\omega \hat{w}_\omega \right) \right) \right) , \quad (82) \]

\[ Q_{\text{wave}}^{\text{total}} = Q_{\text{wave}}^{\text{ideal}} + Q_{\text{wave}}^{\text{diss}} . \quad (83) \]

The total balance of the energy flux is shown in Figure 6. The non-ideal wind momentum flux
Figure 6: Conservation of the total energy flux (in J/m² s) according to Eq. (72) and Eq. (74)-(77). The dark area corresponds to the wave flux, while the light one is the flux of the ideal wind. The dissipative wind flux is negligible compared with the other fluxes and is hidden in the line between them. The total energy flux remains constant, as it can be better seen in Figure 4c.

$$\tilde{P} = P_{\text{nonideal}}^{\text{wind}} = P_{\text{diss}}^{\text{wind}} + P_{\text{total}}^{\text{wave}} = P - P_{\text{ideal}}^{\text{wind}} \quad (84)$$

$$= -\left(\frac{4}{3}\eta + \zeta\right) d\mathcal{U} + \sum_{\omega > 0} W_{\omega} \frac{1}{4} |\hat{b}_{\omega}|^2,$$

$$P_{\text{total}}^{\text{wave}} = P_{\text{ideal}}^{\text{wave}} + P_{\text{diss}}^{\text{wave}} \quad (85)$$
Then according to Eq. (29) we have

\[
\chi(\tau) \equiv \frac{Q^0_{\text{ideal wind}}}{\tau} = Q^0_{\text{ideal wind}}, \tag{86}
\]

\[
\tau(\tau) \equiv \frac{P^0_{\text{ideal wind}}}{\tau} = P^0_{\text{ideal wind}}. \tag{87}
\]

3 Self-Consistent Procedure and Results

First we fix the boundary condition the temperature \(T_0\) and proton density \(n_{p}(0)\) for \(x = 0\). For these parameters we calculate density \(\rho_0\) Debye radius length \(r_D(0)\), Coulomb logarithm \(\Lambda_0\), viscosity \(\eta_0\), heat conductivity \(\kappa_0\) Ohmic resistivity \(\varrho_0\), and sound speed \(c_s(0)\). Initial velocity of the wind is better to be parameterized by the dimensionless parameter \(s \gg 1\), i.e. \(U_0 = c_s(0)/s\). Analogously plasma beta parameter \(\beta_0\) determines the Alfvén speed at \(V_A(0) = \sqrt{\frac{\gamma_2}{2}} c_s(0)\).

Let us also fix the maximal frequency for which we will consider plasma waves \(\omega\) and calculate the absorption rate of the energy density of Alfvén waves \(2k''(0)\). One can choose the interval of the solution of MHD equations to be much larger than the AW mean free path \(1/2k''(0)\), for example

\[
l = \frac{10}{2k''(0)} = \frac{10V_A^2(0)}{\nu_k(0)\omega^2}. \tag{88}
\]

Having units for length \(l\), velocity \(U_0\) and density \(\rho_0\) we can calculate dimensionless variables at \(\tau = 0\) : \(\varpi, \eta_0, \) and \(\Theta_0\).

The input parameters of the program are \(T_0, n_{\text{tot}}(0), \beta_0, \omega, \) and \(s\) which parameterizes \(j = \langle m \rangle n_{\text{tot}}(0)/U_0\). We calculate \(l, a, \eta_0 = \varpi, \kappa_0\) and choose some \(A_{\text{wave}}\) which finally determines increasing of the temperature \(T(1) = T(l)/T(0)\).

The first step in the self-consistent procedure is to choose some initial approximations for the wind variables, the simplest possible functions are \(\overline{T}(\tau) = \overline{U}(\tau) = 1\) and of course \(W(\tau) = F(\tau) = 0\), i.e. to extend the boundary conditions at \(\tau = 0\) for the whole interval \(\tau \in (0, 1)\). Then one can start the successive approximations.

1. At fixed wind profiles \(\overline{T}(\tau)\) and \(\overline{U}(\tau)\) we calculate \(\overline{W}, \overline{F}, \Lambda = \Lambda_0 + \frac{3}{2} \ln \overline{T} + \frac{1}{2} \ln \overline{U}, \varpi, \overline{\eta}, \overline{\kappa}, \) and dimensionless matrices \(\overline{M}\) and \(\overline{H}\). Then we have to solve the wave equation, and to renormalize the solution with some fixed dimensionless energy flux for the R-mode \(Q_{\text{wave}}(0)\).

2. Using so obtained wave variables \(\Psi\) we have to solve equations Eq. (31) and Eq. (32) to find \(\overline{U}(\tau)\) and \(\overline{H}(\tau)\), and respectively \(\overline{T}(\tau)\) from Eq. (30). The variables \(\varpi, \overline{\eta}, \overline{d} \overline{\eta}, \) and \(\overline{d} \overline{\kappa}\), which participate in the coefficients have to be calculated simultaneously.
(a) Temperature profile of the solar wind found by application of the self-consistent procedure described in Section 3.

(b) Derivative of the temperature.

(c) Relative difference between two consecutive solutions for the wind temperature. $T_{old}(x)$ is the temperature calculated during the previous execution of the self-consistent procedure. The relative difference is a measure of the accuracy of the solution. When we approach the actual solution, this difference becomes 0.

Figure 7: Temperature profile derived from the solution of the wind Eq. (31) and (32).
3. Having solved the equations for the wind variables and formerly the equations for the wave variables we can calculate the total energy and momentum flux. If these fluxes are not constant with the required accuracy we go to item 1 and repeat the procedure.

The results from the self-consistent procedure for the temperature profile of the solar wind are shown in Figure 7. Now we can calculate the width of the transition layer by the maximal value of the logarithmic derivative of the temperature

$$\frac{l}{\lambda} = \max_\sigma |d\ln T|.$$  \hspace{1cm} (89)

The width of the transition layer $\lambda$, and the increasing of the temperature $T(1)$ are functions of the wave energy flux coming from the chromosphere $q_{\text{wave}}(0)$. We have to repeat the calculations for different $A_{\text{wave}}$ in order to derive the required temperature enhancement $T(1; A_{\text{wave}}) = T(l)/T(0)$. At fixed enhancement and chosen spectral density of the AW we have no more free parameters and have to compare the calculations with the other model for heating of the solar corona.

4 Conclusions

In spite that iron was suspicious from the very beginning the problem of Coronium was a 70 year standing mystery until unambiguous identification as Fe$^{13+}$ by Grotrian and Edlen in 1939. The same 70 year time quantum was repeated. In 1947 Alfvén [2] advocated the idea that absorption of AW is the mechanism of heating of solar corona. Unfortunately the idea by Swedish iconoclast [5] has never been realized in original form: what can be calculated, what is measured, what is explained and what is predicted. That is why there is a calamity of ideas still on the arena, for a contemporary review see the SOHO proceedings [6]. From qualitative point of view the narrow width of the transition layer $w = \min |dx/d\ln T(x)|$ is the main property which should be compared against the predictions of other scenarios. For example, in order nanoflare hypothesis to be vindicated ([12]) such reconnections need to explain the narrow width of the transition layer at the same boundary conditions of wind velocity and temperature. Moreover electric fields of the reconnections heat mainly the electron component of the plasma. How then the proton temperature in the corona is higher? Launching of Hinode gave a lot of hints for the existence of AW in the corona ([13]), see also ([14]). However most of these researches were in UV region when high frequency AW which heated are already absorbed. All observations are for low frequency (mHz range) AW for which hot corona is transparent. The best that could be done is to extract low frequency behavior of the spectral density of AW and to extrapolate to higher frequencies responsible for heating. So observed AW are irrelevant for the heating. In order to identify
AW responsible for the heating it is necessary to investigate high frequency (1 Hz range) AW in the cold chromosphere using optical, not UV spectral lines. We are unaware whether such type of experiments are planned. One of the purposes of the present work is to focus the attention of experimentalists on the 1 Hz range AW in the chromosphere, which we predict on the basis of our MHD analysis. For such purposes we suggest Doppler tomography ([11]) of Hα to be used; Ca lines are another possibility. Doppler tomography was successfully used for investigation of rotating objects, such as accretion disks ([15]) and solar tornados ([10]). Here we wish to mark also the Doppler tomography by Coronal Multi-channel Polarimeter build by Tomczyk ([16]). For investigation of AW by Doppler tomography we suggest development of frequency dependent Doppler tomography operating as a lock-in voltmeter. The date from every space pixel should be multiplied by \(\sin \omega t\) and integrated for many wave periods. Finally one can observe time averaged distribution of the AW amplitude. Systematic investigation of such frequency dependent Doppler tomograms will reveal that Swedish iconoclast ([5]) is again right that AW heat the solar corona, after another 70 years of dramatic launching of of vast variety of ideas.

Authors are thankful to Yana Maneva ([19–22]) and Martin Stoev for the collaboration in the early stages of the present research ([9]) when the idea of self-induced opacity was advocated and consideration of many problems related to physics of solar corona. Fruitful comments and discussions with Tsvetan Sariisky are highly appreciated. Authors are also thankful to Eckart Marsch for the hospitality during the conference "From the Heliosphere into the Sun – Sailing Against the Wind" (http://www.mps.mpg.de/meetings/hcor/) and to Steven R. Cranmer, Leon Ofman, Jaime Araneda, Yurii Dumin and Yana Maneva for the interest and comments.

References


[2] H. Alfvén, Granulation, magnetohydrodynamic waves, and the heating of the solar corona, MNRAS 107, 211–219 (1947); In this paper Alfvén suggested the heating mechanism of the solar corona. We note that this pioneeric work regrettably is practically never cited in the contemporary reviews on solar corona heating and launching of the solar wind.


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