Bulg. J. Phys. **38** (2011) 3–28

*This paper is dedicated to the memory of **Matey Mateev**, a missing friend*

Clifford Algebras and Spinors*

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Received *9 March 2011*

Abstract. Expository notes on Clifford algebras and spinors with a detailed discussion of Majorana, Weyl, and Dirac spinors. The paper is meant as a review of background material, needed, in particular, in now fashionable theoretical speculations on neutrino masses. It has a more mathematical flavour than the over twenty-six-year-old *Introduction to Majorana masses* [1] and includes historical notes and biographical data on past participants in the story.

PACS codes: 03.65.Fd, 04.20.Gz

1 Quaternions, Grassmann and Clifford Algebras

Clifford's[1] paper [2] on "geometric algebra" (published a year before his death) had two sources: Grassmann's[2] algebra and Hamilton's[3] quaternions whose

*Lectures presented at the University of Sofia in October-November, 2010. Lecture notes prepared together with **Dimitar Nedanovski** (e-mail: dnedanovski@inrne.bas.bg) and completed during the stay of the author at the ICTP, Trieste and at CERN, Geneva.

[1]William Kingdon Clifford (1845-1879) early appreciated the work of Lobachevsky and Riemann; he was the first to translate into English Riemann's inaugural lecture *On the hypotheses which lie at the bases of geometry*. His view of the physical world as *variation of curvature of space* anticipated Einstein's general theory of relativity. He died (before reaching 34) of tuberculosis, aggravated (if not caused) by overwork.

[2]Hermann Günter Grassmann (1809-1877), a German polymath, first published his fundamental work that led the foundations of linear algebra (and contained the definition of *exterior product*), in 1844. He was too far ahead of his time to be understood by his contemporaries. Unable to get a position as a professor in mathematics, Grassmann turned to linguistic. His sound law of Indo-European (in particular, of Greek and Sanskrit) languages was recognized during his lifetime.

[3]William Rowan Hamilton (1805-1865) introduced during 1827-1835 what is now called *Hamiltonian* but also the Lagrangian formalism unifying mechanics and optics. He realized by that time that multiplication by a complex number of absolute value one is equivalent to a rotation in the Euclidean (complex) 2-plane $\mathbb{C}$ and started looking for a 3-dimensional generalization of the complex numbers that would play a similar role in the geometry of 3-space. After many unsuccessful attempts, on October 16, 1843, while walking along the Royal Canal, he suddenly had the inspiration that not three but a four dimensional generalization of $\mathbb{C}$ existed and was doing the job – see introduction to [3].

three imaginary units i, j, k could be characterized by

$$i^2 = j^2 = k^2 = ijk = -1. \quad (1)$$

We leave it to the reader to verify that these equations imply $ij = k = -ji, jk = i = -kj, ki = j = -ik$.

We proceed to the definition of a (real) Clifford algebra and will then display the Grassmann and the quaternion algebras as special cases.

Let V be a real vector space equipped with a quadratic form $Q(v)$ which gives rise – via *polarization* – to a symmetric bilinear form B, such that $2B(u,v) = Q(u+v) - Q(u) - Q(v)$. The *Clifford algebra* $Cl(V,Q)$ is the associative algebra freely generating by V modulo the relations

$$v^2 = Q(v)(= B(v,v)) \text{ for all } v \in V\ ,\ \Leftrightarrow uv + vu = 2B(u,v) \equiv 2(u,v). \quad (2)$$

(Here and in what follows we identify the vector $v \in V$ with its image, say, $i(v)$ in $Cl(V,Q)$ and omit the symbol 1 for the algebra unit on the right hand side.) In the special case $B = 0$ this is the *exterior* or *Grassmann* algebra $\Lambda(V)$, the direct sum of skewsymmetric tensor products of $V = \mathbb{R}^n$

$$\Lambda(V) = \oplus_{k=0}^{n} \Lambda^k(V) \ \Rightarrow\ dim\Lambda(V) = \sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n. \quad (3)$$

Having in mind applications to the algebra of γ-matrices we shall be interested in the opposite case in which B is a non-degenerate, in general indefinite, real symmetric form

$$Q(v) = (v,v) = v_1^2 + ... + v_p^2 - v_{p+1}^2 - ... - v_n^2\,, \quad n = p+q. \quad (4)$$

We shall then write $Cl(V,Q) = Cl(p,q)$, using the shorthand notation $Cl(n,0) = Cl(n),\ Cl(0,n) = Cl(-n)$ in the Euclidean (positive or negative definite) case.[4] The expansion (3) is applicable to an arbitrary Clifford algebra providing a $\mathbb{Z}$ *grading* for any $Cl(V) \equiv Cl(V,Q)$ *as a vector space* (not as an algebra). To see this we start with a basis $e_1, ..., e_n$ of orthogonal vectors of V and define a linear basis of $Cl(V)$ by the sequence

$$\begin{aligned} &1, ..., (e_{i_1} ... e_{i_k}\,,\ 1 \le i_1 < i_2 < ... < i_k \le n),\ k = 1, 2, ..., n \\ &(2e_ie_j = [e_i, e_j] \text{ for } i < j). \end{aligned} \quad (5)$$

It follows that the dimension of $Cl(p,q)$ is again $2^n(n = p+q)$. We leave it as an exercise to the reader to prove that $Cl(0) = \mathbb{R},\ Cl(-1) = \mathbb{C},\ Cl(-2) = \mathbb{H}$,

[4] Mathematicians often use the opposite sign convention corresponding to $Cl(n) = Cl(0,n)$ that fits the case of normed star algebras – see [3] which contains, in particular, a succinct survey of Clifford algebras in Section 2.3. The textbook [4] and the (46-page-long, mathematical) tutorial on the subject [5] use the same sign convention as ours but opposite to the monograph [6]. The last two references rely on the modern classic on Clifford modules [7].

where $\mathbb{H}$ is the algebra of quaternions; $Cl(-3) = \mathbb{H} \oplus \mathbb{H}$. (*Hint*: if e_ν form an orthonormal basis in V (so that $e_\nu^2 = -1$) then in the third case, set $e_1 = i, e_2 = j, e_1 e_2 = k$ and verify the basic relations (1); verify that in the fourth case the operators ${}^1/_2(1 \pm e_1 e_2 e_3)$ play the role of orthogonal projectors to the two copies of the quaternions.) An instructive example of the opposite type is provided by the algebra $Cl(2)$. If we represent in this case the basic vectors by the real 2×2 Pauli matrices $e_1 = \sigma_1, e_2 = \sigma_3$, we find that $Cl(2)$ is isomorphic to $\mathbb{R}[2]$, the algebra of all real 2×2 matrices. If instead we set $e_2 = \sigma_2$ we shall have another algebra (over the real numbers) of complex 2×2 matrices. An invariant way to characterize $Cl(2)$ (which embraces the above two realizations) is to say that it is isomorphic to the complex 2×2 matrices invariant under an $\mathbb{R}$-linear involution given by the complex conjugation K composed with an inner automorphism. In the first case the involution is just the complex conjugation; in the second it is K combined with a similarity transformation: $x \to \sigma_1 K x \sigma_1$.

We note that $Cl(-n), n = 0, 1, 2$ are the only *division rings* among the Clifford algebras. All others have *zero divisors*. For instance, $(1+e_1 e_2 e_3)(1-e_1 e_2 e_3) = 0$ in $Cl(-3)$ albeit none of the two factors is zero.

Clifford algebras are $\mathbb{Z}_2$ *graded*, thus providing an example of *superalgebras*. Indeed, the linear map $v \to -v$ on V which preserves $Q(v)$ gives rise to an involutive automorphism α of $Cl(V, Q)$. As $\alpha^2 = id$ (the identity automorphism) - the defining property of an involution - it has two eigenvalues, ± 1; hence $Cl(V)$ splits into a direct sum of *even* and *odd* elements

$$Cl(V) = Cl^0(V) \oplus Cl^1(V), Cl^i(V) = \oplus_{k=0}^{[n/2]} \Lambda^{i+2k} V,\ i = 0, 1. \tag{6}$$

Exercise 1.1 Demonstrate that $Cl^0(V, Q)$ is a Clifford subalgebra of $Cl(V, Q)$; more precisely, prove that if V is the orthogonal direct sum of a 1-dimensional subspace of vectors collinear with v and a subspace U then $Cl^0(V, Q) = Cl(U, -Q(v)Q|_U)$, where $Q|_U$ stands for restriction of the form Q to U. Deduce that, in particular,

$$Cl^0(p, q) \simeq Cl(p, q-1) \text{ for } q > 0\,, \quad Cl^0(p, q) \simeq Cl(q, p-1) \text{ for } p > 0\,. \tag{7}$$

In particular, for the algebra $Cl(3, 1)$ of Dirac[5] γ-matrices the even subalgebra (which contains the generators of the Lorentz Lie algebra) is isomorphic to $Cl(3) \simeq Cl(1, 2)$ (isomorphic as algebras, not as superalgebras: their gradings are inequivalent).

We shall reproduce without proofs the classification of real Clifford algebras. (The examples of interest will be treated in detail later on.) If R is a ring, we denote by $R[n]$ the algebra of $n \times n$ matrices with entries in R.

[5]Paul Dirac (1902-1984) discovered his equation (the "square root" of the d'Alembert operator) describing the electron and predicting the positron in 1928 [8]. He was awarded for it the Nobel Prize in Physics in 1933. His quiet life and strange character are featured in a widely acclaimed biography [9].

Proposition 1.1 *The following symmetry relations hold*

$$Cl(p+1, q+1) = Cl(p, q)[2], \; Cl(p+4, q) = Cl(p, q+4). \tag{8}$$

They imply the Cartan-Bott[6] *periodicity theorem*

$$Cl(p{+}8, q) = Cl(p{+}4, q{+}4) = Cl(p, q{+}8) = Cl(p, q)[16] = Cl(p, q)\otimes\mathbb{R}[16]. \tag{9}$$

Let $(e_1, ..., e_p, e_{p+1}, ..., e_n), n = p + q$ be an orthonormal basis in V, so that

$$(e_i, e_j)(= B(e_i, e_j)) = \eta_{ij} := e_i^2\delta_{ij,}, \; e_1^2 = ... = e_p^2 = ... = -e_n^2 = 1. \tag{10}$$

Define the *(pseudoscalar) Coxeter*[7] *"volume" element*

$$\omega = e_1 e_2 ... e_n \;\Rightarrow\; \omega^2 = (-1)^{(p-q)(p-q-1)/2}. \tag{11}$$

Proposition 1.2 *The types of algebra $Cl(p, q)$ depend on $p - q \mod 8$ as displayed in Table 1*

Table 1.

$p-q \mod 8$	ω^2	*Cl*(p,q) $p+q=2m$	$p-q \mod 8$	ω^2	*Cl*(p,q) $p+q=2m+1$
0	$+$	$\mathbb{R}[2^m]$	1	$+$	$\mathbb{R}[2^m] \oplus \mathbb{R}[2^m]$
2	$-$	$\mathbb{R}[2^m]$	3	$-$	$\mathbb{C}[2^m]$
4	$+$	$\mathbb{H}[2^{m-1}]$	5	$+$	$\mathbb{H}[2^{m-1}] \oplus \mathbb{H}[2^{m-1}]$
6	$-$	$\mathbb{H}[2^{m-1}]$	7	$-$	$\mathbb{C}[2^m]$

The reader should note the appearance of a complex matrix algebra in two of the above realizations of $Cl(p, q)$ for odd dimensional real vector spaces. The algebra $Cl(4, 1) = \mathbb{C}[4](= Cl(2, 3))$ is of particular interest: it appears as an extension of the Lorentz Clifford algebra $Cl(3, 1)$ (as well as of $Cl(1, 3)$). As we shall see later (see Proposition 2.2, below) $Cl(4, 1)$ gives rise in a natural way to the central extension $U(2, 2)$ of the spinorial conformal group and of its Lie algebra.

Exercise 1.2 Prove that for $n(= p + q)$ odd the Coxeter element of the algebra $Cl(p, q)$ is central and defines a complex structure for $p - q = 3$ mod 4. For n even its $\mathbb{Z}_2$-graded commutator with homogeneous elements vanishes

$$\omega x_j = (-1)^{j(n-1)} x_j \omega \; \text{for} j = 0, 1. \tag{12}$$

[6]*Élie Cartan* (1869-1951) developed the theory of Lie groups and of (antisymmetric) differential forms. He discovered the 'period' 8 in 1908 - see [3, 10], where the original papers are cited. *Raoul Bott* (1923-2005) established his version of the periodicity theorem in the context of homotopy theory of Lie groups in 1956 - see [11] and references therein.

[7](H.S.M.) Donald Coxeter (1907-2003) was born in London, but worked for 60 years at the University of Toronto in Canada. An accomplished pianist, he felt that mathematics and music were intimately related. He studied the product of reflections in 1951.

For proofs and more details on the classification of Clifford algebras – see [4], Section 16, or [6] (Chapter I, Section 4), where also a better digested "Clifford chessboard" can be found (on p. 29). The classification for $q = 0, 1$ can be extracted from the matrix representation of the Clifford units, given in Section 3.

Historical note. The work of Hamilton on quaternions was appreciated and continued by Arthur Cayley (1821-1895), "the greatest English mathematician of the last century – and this", in the words of H.W. Turnbull (of 1923) [12]. Cayley rediscovered (after J.T. Graves) the octonions in 1845. Inspired and supported by Cayley in his student years, Clifford defined his *geometric algebra* [2] (discovered in 1878) as generated by n orthogonal unit vectors, $\mathbf{e}_1, ..., \mathbf{e}_n$, which anticommute, $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ (like in Grassmann) and satisfy $\mathbf{e}_i^2 = -1$ (like in Hamilton), both preceding papers appearing in 1844 (on the eve of Clifford's birth). In a subsequent article, published posthumously, in 1882, Clifford also considered the algebra $Cl(n)$ with $\mathbf{e}_i^2 = 1$ for all i. He distinguished four classes of geometric algebras according to two sign factors: the square of the Coxeter element (11) and the factor $(-1)^{n-1}$ appearing in $\omega e_i = (-1)^{n-1} e_i \omega$ (cf. Eq. (12)). It was Élie Cartan who identified in 1908 the general Clifford algebras $Cl(p, q)$ with matrix algebras with entries in $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and found the period 8 as displayed in Table 1. A nostalgic survey of quaterions and their possible applications to physics is contained in the popular article [13]. A lively historical account of Clifford algebras and spinors is given by Andrzej Trautman - see, in particular, the first reference [14] as well as in his book [15], written jointly with Paolo Budinich - the physicist who was instrumental in founding both the ICTP and the International School for Advanced Studies (SISSA-ISAS) in Trieste, and is a great enthusiast of Cartan spinors.

2 The Groups $Pin(p, q)$ and $Spin(p, q)$; Conjugation and Norm

Define the unique antihomomorphism $x \to x^\dagger$ of $Cl(V)$ called the *conjugation* for which

$$v^\dagger = -v \text{ for all } v \in V \text{ (and } (xy)^\dagger = y^\dagger x^\dagger \text{ for } x, y \in Cl(V)). \tag{13}$$

Whenever we consider a complexification of our Clifford algebra we will extend this antihomomorphism to an antilinear antiinvolution (that is, we assume that $(cx)^\dagger = \bar{c}x^\dagger$ for any $c \in \mathbb{C}, x \in Cl(V)$, where the bar stands for complex conjugation. We shall say that an element $x \in Cl(V)$ is *pseudo(anti)hermitean* if $x^\dagger = (-)x$. The notion of conjugation allows to define a map

$$N : Cl(V) \longrightarrow Cl(V),\ N(x) = xx^\dagger, \tag{14}$$

called *norm*. It extends, in a sense, the quadratic form $-Q$ to the full Clifford algebra and coincides with the usual (positive) "norm squared" on the quaternions

$$N(s + x\mathbf{i} + y\mathbf{j} + z\mathbf{ij}) = s^2 + x^2 + y^2 + z^2 \text{ for } s + x\mathbf{i} + y\mathbf{j} + z\mathbf{ij} \in Cl(-2).$$

For products of vectors of $V, N(x)$ is a scalar: one easily verifies the implication

$$x = v_1 ... v_k \Rightarrow xx^\dagger = (-1)^k Q(v_1)...Q(v_k)(= N(v_1...v_k)). \qquad (15)$$

This would suffice to define the groups $Pin(n)$ and $Spin(n)$ as products of Clifford units (cf. Section 2.4 of [3]). We shall sketch here the more general approach of [7] and [6](digested in the "tutorial on Clifford algebra and the groups **Spin** and **Pin**" [5]).

Let $Cl(p,q)^*$ be the group of invertible elements of $Cl(p,q)$. It seems natural to use its *adjoint action* on $V, \mathrm{Ad_x} v := xvx^{-1}$, to define a covering of the ortrhogonal group $O(p,q)$ as it automatically preserves the quadratic form (4): $(xvx^{-1})^2 = v^2$ (provided $x \in Cl(p,q)^*$ is such that $\mathrm{Ad_x} v \in V$ for all $v \in V$). The adjoint action, however, does not contain the reflections

$$-uvu^{-1} = v - 2\frac{(u,v)u}{u^2}, \text{ for } u \in V,\ u^{-1} = u/u^2, \qquad (16)$$

for an odd dimensional V. To amend this we shall use, following [7] and [6], a *twisted adjoint* representation. We define the *Clifford* (or *Lipschitz*[8]*) group* $\Gamma_{p.q}$ through its action on $V = \mathbb{R}^{p,q}$:

$$x \in \Gamma_{p,q} \text{ iff } \rho_x : v \to \alpha(x)vx^{-1} \in V, \text{ for any } v \in V, \qquad (17)$$

where α is the involutive automorphism which maps each odd element $x \in Cl^1(V)$ (in particular, each element in V) to $-x$ (the involution α was, in fact, used in Section 1 to define the $\mathbb{Z}_2$-grading on $Cl(p,q)$). It is not obvious that the map (17) preserves the form $Q(v) = v^2$ (4) since $\alpha(x) \neq \pm x$, for inhomogeneous $x \in \Gamma_{p,q}$. The following theorem verifies it and gives a more precise picture.

Theorem 2.1 *The map $\rho : \Gamma_{p,q} \to O(p,q)$ is a surjective homomorphism whose kernel is the multiplicative group $\mathbb{R}^*1$ of the nonzero scalar multiples of the Clifford unit. The restriction of $N(x)$ to $\Gamma_{p,q}$ is a nonzero scalar.*

In other words, every element (including reflections) of $O(p,q)$ is the image (under (17)) of some element $x \in \Gamma_{p,q}$, and, furthermore, if x satisfies $\alpha(x)v = vx$ for all $v \in V$, then x is a real number (times the Clifford unit).

Exercise 2.1 Prove the Theorem for $\Gamma_2 \subset Cl(2)^*$. (*Hint*: prove that $x = a + b^1e_1 + b^2e_2 + ce_1e_2 \in \Gamma_2$ iff either $b^1 = b^2 = 0\,(N(x) = a^2 + c^2 > 0)$ or $a = c = 0\,(N(x) = -(b^1)^2 - (b^2)^2 < 0)$; prove that if x satisfies $\alpha(x)v = vx$ for any $v = v^1e_1 + v^2e_2$ then it is a (real) scalar.)

For a pedagogical proof of Theorem 2.1 - see [6] (Chapter I, Section 2) or [5] (Lemma 1.7 and Proposition 1.8).

[8]The German mathematician Rudolf Lipschitz (1832-1903) discovered independently the Clifford algebras in 1880 and introduced the groups $\Gamma_{0,n}$ – see the appendix *A history of Clifford algebras* in [4]

The group $Pin(p,q)$ is defined as the subgroup of $\Gamma_{p,q}$ of elements x for which $N(x) = \pm 1$. The restriction of the map ρ to $Pin(p,q)$ gives rise to a (two-to-one) homomorphism of $Pin(p,q)$ on the orthogonal group $O(p,q)$. The group $Spin(p,q)$ is obtained as the intersection of $Pin(p,q)$ with the even subalgebra $Cl^0(p,q)$.

For any vector v in $V \subset Cl(p,q)$ each element x of $Spin(p,q)$ defines a map preserving $Q(v)$ (we note that for $x \in Spin(p,q), \alpha(x) = x$, so that the twisted adjoint coincides with the standard one)

$$v \to xvx^{-1} \quad (x^{-1} = N(x)x^{\dagger}, \, \text{for} N(x)^2 = 1). \tag{18}$$

The (connected) group $Spin(p,q)$ can be defined as the double cover of the identity component $SO_0(p,q)$ of $SO(p,q)$ and is mapped onto it under (18). The Lie algebra $spin(p,q)$ of the Lie group $Spin(p,q)$ is generated by the commutators $[e_i, e_j]$ of a basis of $V = \mathbb{R}^{(p,q)}$.

Remark 2.1 Another way to approach the spin groups starts with the observation that the (connected) orthogonal group $SO(n)$ is not simply connected, its *fundamental (or homotopy) group*[9] consists of two elements, $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n > 2$, while for the circle, $n = 2$, it is infinite: $\pi_1(SO(2) = \mathbb{Z}$. The homotopy group of the pseudo-orthogonal group $SO(p,q)$ is equal to that of its maximal compact subgroup

$$\pi_1(SO_0(p,q)) = \pi_1(SO(p)) \times \pi_1(SO(q)) (= \mathbb{Z}_2 \, \text{for} \; p > 2, \, q \leq 1). \tag{19}$$

In all cases the group $Spin(p,q)$ can be defined as the double cover of $SO_0(p,q)$ (which coincides with its universal cover for $p > 2, q \leq 1$).

Exercise 2.2 Verify that the Coxeter element ω (11) generates the centre of $Spin(p,q)$ for $p - q \neq 4$ mod 8 while the centre of $Spin(4\ell)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see Appendix A1 to [16]).

We proceed to describe the *spinor representations*[10] in low dimensions. More precisely, we shall identify $spin(p,q)$ and $Spin(p,q)$ as a sub-Lie-algebra and a subgroup in $Cl(p,q)$. As it is clear from Table 1 for $n (= p + q) = 2m$ there is a single irreducible *Clifford module* of dimension 2^m; for $n = 2m + 1$ there may be two irreducible representations of the same dimension. In either case, knowing the embedding of the spin group into the Clifford algebra we can thereby find its defining representation.

[9] Anticipated by Bernhard Riemann (1826-1866), the notion of fundamental group was introduced by Henri Poincaré (1854-1912)in his article *Analysis Situs* in 1895.

[10] The theory of finite dimensional irreducible representations of (semi)simple Lie groups (including the spinors) was founded by E. Cartan in 1913 - see the historical survey [10]. The word *spinor* was introduced by Paul Ehrenfest (1880-1933) who asked in the fall of 1928 the Dutch mathematician B.L. van der Waerden (1903-1996) to help clear up what he called the “group plague” (see [17] and Lecture 7 in [18]).

Consider the 8-dimensional Clifford algebra $Cl(3)$ spanned by the unit scalar, 1, the three orthogonal unit vectors, $\sigma_j, j = 1,2,3$, the unit bivectors $\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1$, and the pseudoscalar $(\omega_3 \equiv) i := \sigma_1\sigma_2\sigma_3$. It is straightforward to show that the conditions $\sigma_j^2 = 1$ and the anticommutativity of σ_j imply

$$(\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_3\sigma_1)^2 = -1 = i^2. \tag{20}$$

(The σ_j here are just the unit vectors in $\mathbb{R}^3$ that generate $Cl(3)$. We do not use the properties of the Pauli matrices which can serve as their representation.) The subalgebra $Cl^0(3)$ spans the 4-dimensional space $Cl(-2) = \mathbb{H}$ of quaternions, thus illustrating the relation (7). It contains a group of unitaries of the form

$$\begin{aligned} U =& cos(\theta/2) - (n_1\sigma_2\sigma_3 + n_2\sigma_3\sigma_1 + n_3\sigma_1\sigma_2) sin(\theta/2) \\ =& cos(\theta/2) - i\mathbf{n}\boldsymbol{\sigma} sin(\theta/2),\ \mathbf{n}^2 = 1,\ \mathbf{n}\boldsymbol{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3, \end{aligned} \tag{21}$$

that is isomorphic to $SU(2)$. Furthermore, the transformation of 3-vectors v given by (18) with $U^{-1} = U^*(= U^\dagger)$, where $\sigma_j^* = \sigma_j, i^* = -i$ represents an $SO(3)$ rotation on angle θ around the axis $\mathbf{n}$. The map $SU(2) \to SO(3)$ thus defined is two-to-one as $U = -1$ corresponds to the identity $SO(3)$ transformation.

The 16-dimensional Euclidean algebra $Cl(4)$ generated by the orthonormal vectors γ_α such that $[\gamma_\alpha, \gamma_\beta]_+ = 2\delta_{\alpha\beta},\ \alpha, \beta = 1,2,3,4$ is isomorphic to $\mathbb{H}[2]$. Its even part is given by the algebra $Cl(-3)$ discussed in Section 1: $Cl^0(4) \simeq Cl(-3) \simeq \mathbb{H} \oplus \mathbb{H}$. The corresponding Lorentzian[11] Clifford algebra $Cl(3,1)$ is generated by the orthonormal elements γ_μ satisfying

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu},\ \mu,\nu = 0,1,2,3,\ (\eta_{\mu\nu}) = diag(-1,1,1,1). \tag{22}$$

According to (7) the even subalgebra $Cl^0(4,1)$ is isomorphic to the above $Cl(4) \simeq \mathbb{H}[2]$ while $Cl^0(3,1) \simeq Cl(3) \simeq \mathbb{C}[2]$. It contains both the generators $\gamma_{\mu\nu} := 1/2[\gamma_\mu, \gamma_\nu]$ of the Lie algebra $spin(3,1)$ and the elements of the spinorial Lorentz group $SL(2,\mathbb{C})$. It is easy to verify that the elements $\gamma_0\gamma_j$ (corresponding to σ_j in $Cl(3)$) have square one while the pseudoscalar (11) $\omega(= \omega_{3,1}) = \gamma_0\gamma_1\gamma_2\gamma_3$ satisfies $\omega^2 = -1$ and

$$\gamma_{12} = \omega\gamma_{03},\ \gamma_{31} = \omega\gamma_{02},\ \gamma_{23} = \omega\gamma_{01}. \tag{23}$$

It follows that every even element of $Cl(3,1)$ can be written in the form

$$z = z^0 + z^j\gamma_{0j},\ z^\mu = x^\mu + \omega y^\mu,\ \mu = 0,...,3,\ x^\mu, y^\mu \in \mathbb{R}, \tag{24}$$

[11] Hendrik Antoon Lorentz (1853-1928) introduced his transformations describing electromagnetic phenomena in the 1890's. He was awarded the Nobel Prize (together with his student Pieter Zeeman (1865-1943)) "for their research into the influence of magnetism upon radiation phenomena".

thus displaying the complex structure generated by the central element ω of $Cl^0(3,1)$ (of square -1). In particular, the Lie algebra $spin(3,1)$ generated by $z^j\gamma_{0j}$ is nothing but $sl(2,\mathbb{C})$. The group $Spin(3,1)$ (a special case of $Spin(p,q)$ defined in the beginning of this section) is isomorphic to $SL(2,\mathbb{C})$, the group of complex 2×2 matrices of determinant one (which appears as the simply connected group with the above Lie algebra).

Proposition 2.2 (a) *The pseudoantihermitean elements $x \in Cl(4,1)$ (satisfying $x^\dagger = -x$) span the 16-dimensional Lie algebra $u(2,2)$. The corresponding Lie group $U(2,2)$ consists of all pseudounitary elements $u \in Cl(4,1), uu^\dagger = 1$. There exists a (unique up to normalization) $U(2,2)$-invariant sesquilinear form $\widetilde{\psi}\psi = \psi^*\beta\psi$ in the space $\mathbb{C}^4$ of 4-component spinors (viewed as a $Cl(4,1)$-module), where the element β of $Cl(4,1)$ intertwines the standard hermitean conjugation $*$ of matrices with the antiinvolution (13)*

$$\begin{aligned}&\gamma_a^*\beta = -\beta\gamma_a, \Rightarrow \gamma_{ab}^*\beta = -\beta\gamma_{ab},\ a,b = 0,1,2,3,4; \Rightarrow x^*\beta = \beta x^\dagger;\\ &u^*\beta = \beta u^{-1} \text{ for } u \in U(2,2).\end{aligned} \tag{25}$$

(b) *The intersection of $U(2,2)$ with $Cl(3,1)$ coincides with the 10-parameter real symplectic group $Sp(4,\mathbb{R}) \simeq Spin(3,2)$ whose Lie algebra $sp(4,\mathbb{R})$ is spanned by $\gamma_{\mu\nu}$ and by the odd elements $\gamma_\mu \in Cl^1(3,1)$. The corresponding symplectic form is expressed in terms of the charge conjugation matrix C, defined in Section 3 below. An element $\Lambda = c_0 + \sum_{j=1}^3 c_j\gamma_{0j}$ of $Cl^0(3,1)$, $c_\nu = a_\nu + \omega b_\nu$, $a_\nu, b_\nu \in \mathbb{R}$ belongs to $Spin(3,1) \subset Spin(3,2)$ iff $N(\Lambda) = c_0^2 - \mathbf{c}^2 = 1$, where $\mathbf{c}^2 = \sum_{i=1}^3 c_i^2$.*
(c) *Space and time reflections are given by the odd elements*

$$\Lambda_s = \gamma^0\ (\Lambda_s^{-1} = \gamma_0 = -\gamma^0),\ \Lambda_t = \gamma^0\omega\ (\Lambda_t^{-1} = \gamma_0\omega). \tag{26}$$

We have, in general, for $\Lambda \in Pin(3,1)$,

$$\Lambda\gamma p\Lambda^{-1} = \gamma L(\Lambda)p,\ p\gamma := p^\mu\gamma_\mu,\ L(\Lambda) \in O(3,1),\ L(-\Lambda) = L(\Lambda). \tag{27}$$

We leave the proof to the reader, only indicating that $u(2,2)$ is spanned by γ_a, γ_{ab}, and by the central element $\omega_{4,1}$ which plays the role of the imaginary unit.

Exercise 2.3 Verify that $\Lambda = exp(\lambda^{\mu\nu}\gamma_{\mu\nu})$, where $(\lambda^{\mu\nu})$ is a skewsymmetric matrix of real numbers, satisfies the last equation (25) and hence belongs to $Spin(3,1)$. How does this expression fit the one in Proposition 2.2 (b)? Prove that $\Lambda \in \Gamma_{3,1}$ iff $N(\Lambda) \in \mathbb{R}^*$. Verify that $c^*\beta = \beta c$ for $c = a + \omega b, c^* = a - \omega b$ and that $\Lambda^{-1} = \Lambda^\dagger$.

The resulting (4-dimensional) representation of $Spin(3,1)$ (unlike that of $Spin(3,2) \simeq Sp(4,\mathbb{R})$) is reducible and splits into two complex conjugate representations, distinguished by the eigenvalues ($\pm i$) of the central element ω of $Cl^0(3,1)$. These are the (left and right) *Weyl spinors*.

Remark 2.2 If we restrict attention to the class of representations for which the Clifford units are either hermitean or antihermitean then the (anti)hermitean units would be exactly those for which $\gamma_\mu^2 = 1(-1)$. Within this class the matrix β, assumed hermitean, is determined up to a sign; we shall choose it as $\beta = i\gamma^0$. This class is only preserved by unitary similarity transformations. By contrast, the implicit definition of the notion of hermitean conjugation contained in Proposition 2.2 (a) is basis independent.

Exercise 2.4 Prove that the Lie algebra $spin(4) \subset Cl^0(4)$ splits into a direct sum of two $su(2)$ Lie algebras. The Coxeter element ω has eigenvalues ± 1 in this case and the idempotents $1/2(1 \pm \omega)$ project on the two copies of $su(2)$ (each of which has a single 2-dimensional irreducible representation).

Remark 2.3 Denote by $cl(p, q)$ the maximal semisimple Lie algebra (under commutation) of $Cl(p, q),\ p + q = n$. The following list of identifications (whose verification is left to the reader) summarizes and completes the examples of this Section:

$$\begin{aligned}
& cl(2) = sl(2,\mathbb{R}) = cl(1,1); \\
& cl(3) = spin(3,1) \simeq sl(2,\mathbb{C}) = cl(1,2), \\
& cl(2,1) = spin(2,1) \oplus spin(2,1) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}); \\
& cl(4) = spin(5,1) \simeq sl(2,\mathbb{H}) = cl(1,3),\ cl(3,1) = spin(3,3) \simeq sl(4,\mathbb{R}); \\
& cl(5) = spin(5,1) \oplus spin(5,1) \simeq sl(2,\mathbb{H}) \oplus sl(2,\mathbb{H}); \\
& cl(4,1) = sl(4,\mathbb{C}) = cl(2,3);\ cl(3,2) = sl(4,\mathbb{R}) \oplus sl(4,\mathbb{R}); \\
& cl(6) = su(6,2) = cl(5,1); \\
& cl(7) = sl(8,\mathbb{C}),\ cl(6,1) = sl(4,\mathbb{H}); \\
& cl(8) = sl(16,\mathbb{R}),\ cl(7,1) = sl(8,\mathbb{H}); \\
& cl(9) \simeq cl^0(9,1) = sl(16,\mathbb{R}) \oplus sl(16,\mathbb{R})\, cl(8,1) = sl(16,\mathbb{R}).
\end{aligned} \tag{28}$$

Here is also a summary of low dimensional Spin groups ($Spin(p,q) \in Cl^0(p,q)$) (see [19], Table 4.1):

$$\begin{aligned}
Spin(1,1) &= \mathbb{R}_{>0},\ Spin(2) = U(1); \\
Spin(2,1) &= SL(2,\mathbb{R}),\ Spin(3) = SU(2); \\
Spin(2,2) &= SL(2,\mathbb{R}) \times SL(2,\mathbb{R}),\ Spin(3,1) = SL(2,\mathbb{C}), \\
& Spin(4) = SU(2) \times SU(2); \\
Spin(3,2) &= Sp(4,\mathbb{R}),\ Spin(4,1) = Sp(1,1;\mathbb{H}),\ Spin(5) = Sp(2,\mathbb{H}); \\
Spin(3,3) &= SL(4,\mathbb{R}),\ Spin(4,2) = SU(2,2), \\
Spin(5,1) &= SL(2,\mathbb{H}),\ Spin(6) = SU(4).
\end{aligned} \tag{29}$$

3 The Dirac γ-matrices in Euclidean and in Minkowski Space

We shall now turn to the familiar among physicists *matrix representation* of the Clifford algebra and use it to characterize in an alternative way the properties $\mathrm{mod}8$ of $Cl(D)$ and $Cl(D-1,1)$, the cases of main interest. As we have seen (see Table 1) if $p-q \neq 1 \bmod 4$, in particular, in all cases of physical interest in which the space-time dimension D is even, $D=2m$, there is a unique irreducible (2^m-dimensional) representation of the associated Clifford algebra. It follows that for such D any two realizations of the γ-matrices are related by a similarity transformation (for $Cl(4)$ this is the content of the *Pauli*[12] *lemma*). We shall use the resulting freedom to display different realizations of the γ-matrices for $D=4$, suitable for different purposes.

It turns out that one can represent the γ-matrices for any D as tensor products of the 2×2 Pauli σ-matrices [21] (cf. [19, 22]) in such a way that the first $2m$ generators of $Cl(2m+2)$ are obtained from those of $Cl(2m)$ by tensor multiplication (on the left) by, say, σ_1. The generators of $Cl(2m+1)$ give rise to a reducible subrepresentation of $Cl(2m+2)$ whose irreducible components can be read off the represetnation of $Cl(2m)$

$$\begin{aligned}
&Cl(1): \{\sigma_1\}; \quad Cl(2(3)): \{\sigma_i,\ i=1,2,(3)\};\\
&Cl(4): \{\gamma_i = \sigma_1\otimes\sigma_i,\ i=1,2,3,\ \gamma_4 = \sigma_2\otimes 1\ ;\\
&Cl(6): \Gamma_\alpha = \sigma_1\otimes\gamma_\alpha,\ \alpha=1,...,5;\ Cl(8): \Gamma^{(8)}_a = \sigma_1\otimes\Gamma_a,\ a=1,...,7;\\
&Cl(10): \Gamma^{(10)}_a = \sigma_1\otimes\Gamma^{(8)}_a, a=1,...,9,\ \text{where}\, \Gamma^{(2m)}_{2m} = \sigma_2\otimes 1^{\otimes(m-1)},\\
&\Gamma^{(2m)}_{2m+1} = \sigma_3\otimes 1^{\otimes(m-1)} = i^{3-m}\omega_{2m-1,1},
\end{aligned} \tag{30}$$

where $1^{\otimes k} = 1\otimes ...\otimes 1$ (k factors), 1 stands for the 2×2 unit matrix. The Clifford algebra $Cl(D-1,1)$ of D-dimensional Minkowski[13] space is obtained by replacing γ_D with

$$\gamma^0 = i\gamma_{2m}\ (=-\gamma_0)\,\mathrm{for\ D=2m,2m+1}. \tag{31}$$

Note that while $\Gamma^{(2m)}_{2m+1}$ is expressed in terms of a product of $\Gamma^{(2m)}_a$, $a\le 2m$, the element $\Gamma^{(2m+1)}_{2m+1}$ is an independent Clifford unit. In particular, we only know that the product ω_3 of $\sigma_i, i=1,2,3$ in $Cl(3)$ is a central element of square -1, while we have the additional relation $\sigma_1\sigma_2 = i\sigma_3$ in $Cl(2)$, in accord with the fact that the real dimension (8) of $Cl(3)$ is twice that of $Cl(2)$. Furthermore, as one can read off Table 1, the Clifford algebra $Cl(5)$ (or, more generally,

[12]Wolfgang Pauli (1900-1958), Nobel Prize in Physics, 1945 (for his exclusion principle), predicted the existence of a neutrino (in a letter "to Lise Meitner (1878-1968) et al." of 1930 – see [20]); he published his lemma about Dirac's matrices in 1936

[13]Hermann Minkowski (1864-1909) introduced his 4-dimensional space-time in 1907, thus completing the special relativity theory of Lorentz, Poincaré and Einstein.

$Cl(q+5,q)$) is reducible so that the matrices $\Gamma_a^{(2m)}$ in (30) for $1 \le a \le 2m+1$ realize just one of the two irreducible components. Furthermore, γ_{2m+1} is proportional to $\omega(p,q), p+q=2m, q=0,1$ but only belongs, for $q=1$, to the complexification of $Cl(p,q)$; for instance,

$$\gamma_5 = i\omega(3,1)(=\sigma_3 \otimes 1). \tag{32}$$

The algebra $Cl(4,1)$, which contains γ_5 as a real element, plays an important role in physical applications that seems to be generally ignored. Its Coxeter element $\omega(4,1)$, being central of square -1, gives rise to a complex structure (justifying the isomorphism $Cl(4,1) = \mathbb{C}[4]$ that can be read off Table 1). The Lie algebra $cl(4,1) = sl(4,\mathbb{C})$ (see Eq. (28)) has a real form $su(2,2) = \{x \in cl(4,1); x^\dagger = -x\}$; the corresponding Lie group is the *spinorial conformal group* $SU(2,2) = \{\Lambda \in Cl(4,1); \Lambda^\dagger = \Lambda^{-1}\}$ which preserves the *pseudohermitean form* $\tilde{\psi}\psi$.

We proceed to defining *(charge) conjugation*, in both the Lorentzian and the Euclidean framework, and its interrelation with γ_{2m+1} for $D=2m$. This will lead us to the concept of KO*-dimension* which provides another mod 8 characteristic of the Clifford algebras. (It has been used in the noncommutative geometry approach to the standard model (see [23–25] for recent reviews and references and [26] for the Lorentzian case).

We define the *charge conjugation matrix* by the condition

$$-\gamma_a^t C = C\gamma_a \tag{33}$$

which implies

$$-\gamma_{ab}^t C = C\gamma_{ab} \; (2\gamma_{ab} = [\gamma_a, \gamma_b]), \tag{34}$$

but

$$\begin{aligned}\gamma_{abc}^t C = C\gamma_{abc} \; (6\gamma_{abc} &= [\gamma_a, [\gamma_b, \gamma_c]]_+ - \gamma_b\gamma_a\gamma_c + \gamma_c\gamma_a\gamma_b \\ &= -6\gamma_{bac} = 6\gamma_{cab} = ...).\end{aligned} \tag{35}$$

(In view of (31), if (33) is satisfied in the Euclidean case, for $\alpha = 1,...,D$, then it also holds in the Lorentzian case, for $\mu = 0,...,D-1$.) It is straightforward to verify that given the representation (30) of the γ-matrices there is a unique, up to a sign, choice of the charge conjugation matrix $C(2m)$ (for an even dimensional space-time) as a product of $Cl(2m-1,1)$ units

$$\begin{aligned}&C(2) = c := i\sigma_2, C(4) = \gamma_3\gamma_1 = 1 \otimes c, C(6) = \Gamma^0\Gamma_2\Gamma_4 = c \otimes \sigma_3 \otimes c,\\ &C(8) = \Gamma_1\Gamma_3\Gamma_5\Gamma_7 = 1 \otimes c \otimes \sigma_3 \otimes c, (\Gamma_a \equiv \Gamma_a^{(8)}),\\ &C(10) = \Gamma^0\Gamma_2\Gamma_4\Gamma_6\Gamma_8 = c \otimes \sigma_3 \otimes c \otimes \sigma_3 \otimes c.\end{aligned} \tag{36}$$

The above expressions can be also used to write down the charge conjugation matrix for odd dimensional space times. A natural way to do it is to embed

$Cl(2m-1)$ into $Cl(2m)$ thus obtaining a reducible representation of the odd Clifford algebras. Then we have two inequivlent solutions of (33)

$$\begin{aligned}
&C(2m-1) := C(2m) \Rightarrow \overline{C}(5)C(5) = \overline{C}(7)C(7) = 1 = -\overline{C}(3)C(3).\\
&C'(2m-1) = i^{5-m}\omega_{2m-1}C(2m)(= -i^{5-m}C(2m)\omega_{2m-1}),\\
&\Rightarrow \overline{C}'(2m-1)C'(2m-1) = -\overline{C}(2m-1)C(2m-1).
\end{aligned} \tag{37}$$

In particular, $C(5)$ and $C'(5)$ (satisfying (33) for $1 \le a \le 5$) only exist in a reducible 8-dimensional representation of $Cl(4,1)(\in Cl(5,1))$. (We observe that, with the above choice of phase factors, all matrices C are real.)

We define (in accord with [23]) the *euclidean charge conjugation operator* as an antiunitary operator J in the 2^m-dimensional complex Hilbert space $\mathcal{H}$ (that is an irreducible Clifford module – *i.e.*, the (spinor) representation space of $Cl(2m)$) expressed in terms of the matrix $C(2m)$ followed by complex conjugation

$$J = KC(2m) \Rightarrow J^2 = \bar{C}(2m)C(2m) = (-1)^{m(m+1)/2}. \tag{38}$$

We stress that Eq. (38) is independent of possible i-factors in C (that would show up if one assumes that $C(2m)$ belongs, *e.g.*, to $Cl(2m)$).

Alain Connes [27] defines the *KO dimension* of the (even dimensional) non-commutative internal space of his version of the standard model by two signs: the sign of J^2 (38) and the factor $\epsilon(m)$ in the commutation relation between $J = J(2m)$ and the *chirality operator* $\gamma := \gamma_{2m+1}$

$$J\gamma = \epsilon(m)\gamma J \ (\gamma = \gamma^*, \gamma^2 = 1). \tag{39}$$

Since γ_{2m+1} of (30) is real the second sign factor is determined by the commutation relation between $C(2m)$ and γ_{2m+1}; one finds

$$\epsilon(m) = (-1)^m. \tag{40}$$

The signature, $(+,-)$, needed in the noncommutative geometry approach to the standad model (see [23]), yields KO dimension 6 mod 8 of the internal space (the same as the dimension of the compact Calabi-Yau manifold appearing in superstring theory).

The charge conjugation operator for Lorentzian spinors involves the matrix β of Eq. (25) that defines an invariant hermitean form in $\mathbb{C}^4$ (multiplied by an arbitray phase factor η which we shall choose to make the matrix $\eta\beta C$ appearing in (41) below real)

$$\begin{aligned}
&J_L = K\eta\beta C \Rightarrow J_L^2 = \bar{B}B \,\text{where}\, B := \eta\beta C(= \gamma^0 C)\\
&\Rightarrow B^t = (= B^*) = B \,\text{for}\, Cl(p,1), p = 1,2,3\,\text{mod}8\\
&B^t = -B\,(B^2 = -1)\,\text{otherwise}.
\end{aligned} \tag{41}$$

It follows that J_L^2 has the opposite sign of J^2. It is easy to verify that $\epsilon(m)$ also changes sign when using the charge conjugation for Lorentzian signature

$$J_L^2 = -J^2,\ \epsilon_L(m) = -\epsilon(m). \tag{42}$$

In both cases the above two signs in a space-time of dimension $2m$ (and hence the KO dimension) is periodic in m of period 4.

Whenever $J^2 = 1$, the charge conjugation allows to define the notion of *real* or *Majorana spinor*. Indeed, in this case J admits the eigenvalue 1 and we shall say that ψ is a Majorana spinor if $J\psi = \psi$. It is clear from Table 1 that Majorana spinors only exist for signatures $p - q = 0, 1, 2$ mod 8 ($p(= D - 1) = 1, 2, 3$ for $Cl(p, 1)$ (41).

Exercise 3.1 Prove that $J\Lambda J = \Lambda$ for $J^2 = 1, \Lambda \in Spin(p, q)$ so that the above reality property is $Spin(p, q)$-invariant.

Since the chirality operator (which only exists in dimension $D = 2m$) has square 1 (according to (39)) it has two eigenspaces spanned by two $2^{\,m-1}$-dimensional *Weyl*[14] *spinors*. They are complex conjugate to each other for $p - q = 2$ mod 4 (*i.e.* for $Cl(2), Cl(3, 1), Cl(6), Cl(7, 1)$); self-conjugate for $p - q = 4$ (for $Cl(4), Cl(5, 1)$; they are (equivalent to) real *Majorana-Weyl spinors* for $p - q = 0$ mod 8 (1-dimensional for $Cl(1, 1)$, 8-dimensional for $Cl(8)$, 16-dimensional for $Cl(9, 1)$).

Consider the simplest example of a Majorana-Weyl field starting with the massless *Dirac equation* in the $Cl(1, 1)$ module of 2-component spinor-valued functions ψ of $x = (x^0, x^1)$

$$\gamma\partial\psi \equiv (\gamma^0\partial_0 + \gamma^1\partial_1)\psi = 0, \tag{43}$$

$$\gamma^0 = c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \gamma^1 = \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \partial_\nu = \frac{\partial}{\partial x^\nu}$$

The chirality operator is diagonal in this basis, so that the two components of ψ can be interpreted as "left and right"

$$\gamma = \gamma^0\gamma^1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ \Rightarrow\ \Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}. \tag{44}$$

Thus equation (43) can be written as a (decoupled!) system of *Weyl equations*:

$$(\partial_0 + \partial_1)\psi_R = 0 = (\partial_1 - \partial_0)\psi_L, \tag{45}$$

implying that the chiral fields behave as a *left* and *right* movers:

$$\psi_L = \psi_L(x^0 + x^1),\ \psi_R = \psi_R(x^0 - x^1). \tag{46}$$

[14] Hermann Weyl (1885-1955) worked in Göttingen, Zürich and Princeton. He came as close as anyone of his generation to the universalism of Henri Poincaré and of his teacher David Hilbert (1862-1943). He introduced the 2-dimensional spinors in $Cl(3, 1)$ for a "massless electron" in [28]; he wrote about spinors in n dimensions in a joint paper with the German-American mathematician Richard Brauer (1901-1977) in 1935 [22].

A priori ψ_c, $c = L, R$ are complex valued functions, but since the coefficients of the Dirac equation are real ψ_c and $\bar{\psi}_c$ satisfy the same equation, in particular, they can be both real. These are the (1-component) Majorana-Weyl fields (appearing, *e.g.*, in the chiral Ising model - see for a review [29]).

Exercise 3.2 Prove that there are no Majorana-Weyl solutions of the Dirac equation $(\sigma_1\partial_1 + \sigma_2\partial_2)\Psi_E = 0$ in the $Cl(2)$ module (E standing for Euclidean), but there is a 2-component Majorana spinor, such that the two components of Ψ_E are complex conjugate to each other.

We are not touching here the notion of *pure spinor* which recently gained popularity in relation to (multidimensional) superstring theory – see [30] for a recent review and [31] for a careful older work involving 4-fermion identities.

Historical note. The enigmatic genius *Ettore Majorana* (1906-1938(?)) has fascinated a number of authors. For a small sample of writngs about him - see (in order of appearance) [32–35], and Appendix A to [36] (where his biography by E. Amaldi in Majorana's collected work is also cited). Let me quote at some length the first hand impressions of Majorana of another member of the"circle of Fermi", Bruno Pontecorvo (for more about whom – see the historical note to Sectiuon 5): "When I joined as a first year student the Physical Institute of the Royal University of Rome (1931) Majorana, at the time 25 years old, was already quite famous within the community of a few Italian physicists and foreign scientists who were spending some time in Rome to work under Fermi. The fame reflected first of all the deep respect and admiration for him of Fermi, of whom I remeber exactly these words: "once a physical question has been posed, no man in the world is capable of answering it better and faster than Majorana". According to the joking lexicon used in the Rome Laboratory, the physicists, pretending to be associated with a religious order, nicknamed the infalliable Fermi as the Pope and the intimidating Majorana as the Great Inquisitor. At seminars he was usually silent but occasionally made sarcastic and paradoxical comments, always to the point. Majorana was permanently unhappy with himself (and not only with himself!). He was a pessimist, but had a very accute sense of humour. It is difficult to imagine persons as different in character as Fermi and Majorana... Majorana was conditioned by complicated ... living rules ... In 1938 he literally disappeared. He probably committed suicide but there is no absolute certainty about this point. He was quite rich and I cannot avoid thinking that his life might not have finished so tragically, should he have been obliged to work for a living." Majorana thought about the neutron before James Chadwick (1891-1974) discovered it (in 1932 and was awarded the Nobel Prize for it in 1935) and proposed the theory of "exchange forces" between the proton and the neutron. Fermi liked the theory but Ettore was only convinced to publish it by Werner Heisenberg (1901-1976) who was just awarded the Nobel Prize in Physics when Majorana visited him in 1933. Majorana was not happy with Dirac's hole theory of antiparticles (cf. the discussion in [37]). In 1932, in a paper "Relativistic theory of particles with arbitrary intrinsic angular momentum" (introducing the first infinite dimensional representation of the Lorentz group) he devised an infinite component wave equation with no antiparticles (but with a continuous tachyonic mass spectrum). His last paper [38] that was, in the words of [32], forty years ahead of its time, is also triggered by this dissatisfaction [15]. In its summary (first translated into English by Pontecorvo) he acknowledges that for electrons and positrons his approach may only lead to a formal progress. But, he concludes "it is perfectly possible to construct in a very natural way a theory of neutral particles without negative (energy) states." The important physical consequence of the (possible) existence of a truly neutral (Majorana) particle - the *neutrinoless double beta decay* - was extracted only one year later, in 1938, by Wendel H. Furry (1907-1984) in what Pontecorvo calls "a typical incubation paper ... stimulated by Majorana and (Giulio) Racah (1909-1965) thinking" and still awaits its experimental test.

4 Dirac, Weyl and Majorana Spinors in 4D Minkowski Space-Time

For a consistent physical interpretation of spinors, one needs local anticommuting (spin 1/2) quantum fields. (Their "classical limit" will produce an object which is unknown in physics: strictly anticommuting Grassmann valued fields.)We choose to build up the complete picture step by step, following roughly, the historical development.

To begin with, the Dirac spinors form a spinor bundle over 4-dimensional space-time with a $\mathbb{C}^4$ fibre.(We speak of elements of a fibre bundle, rather than functions on Minkowski space, since $\psi(x)$ is double valued: it changes sign under rotation by 2π.) The spinors span an irreducible representation (IR) of $Cl(3,1)$ which remains irreducible when restricted to the group $Pin(3,1)$, but splits into two inequivalent IRs of its connected subgroup $Spin(3,1) \simeq SL(2,\mathbb{C})$. These IRs are spanned by the 2-component "left and right" Weyl spinors, eigenvectors of the chirality

$$(\gamma =)\gamma_5 = i\omega_{3,1} = i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{47}$$

Remark 4.1 Relativistic local fields transform under finite dimensional representations of $SL(2,\mathbb{C})$, the *quantum mechanical Lorentz group* – see Section 5.6 of [39] for a description of these representations targeted at applications to the theory of quantum fields. Here we just note that the finite dimensional irreducible representations (IRs) of $SL(2,\mathbb{C})$ are labeled by a pair of half-integer numbers $(j_1, j_2), j_i = 0, 1/2, 1, \ldots$. Each IR is spanned by *spin-tensors* $\Phi_{A_1\ldots A_{2j_1}\dot{B}_1\ldots\dot{B}_{2j_2}}, A, \dot{B} = 1,2$, symmetric with respect to the

[15] According to the words, which A. Zichichi [34] ascribes to Pontecorvo, it was Fermi who, aware of Majorana's reluctance to write up what he has done, wrote himself the article, after Majorana explained his work to him.

dotted and undotted indices, separately; thus the dimension of such an IR is $dim(j_1, j_2) = (2j_1+1)(2j_2+1)$. The Weyl spinors ψ_L and ψ_R, introduced below, transform under the basic (smallest nontrivial) IRs $(1/2, 0)$ and $(0, 1/2)$ of $SL(2, \mathbb{C})$, respectively. Their direct sum span the 4-dimensional Dirac spinors which transform under an IR of $Pin(3, 1)$ (space reflection exchanging the two chiral spinors).

The "achingly beautiful" (in the words of Frank Wilczek, cited in [9], p.142) Dirac equation [8] for a free particle of mass m, carved on Dirac's commemorative stone in Westminster Abbey, has the form[16]

$$(m + \gamma\partial)\psi = 0,\ \gamma\partial = \gamma^\mu\partial_\mu,\ \psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}, \widetilde{\psi}(m - \gamma\partial) = 0 \text{ for } \widetilde{\psi} = \psi^*\beta$$

(the partial derivatives in the equation for $\widetilde{\psi}$ acting to the left). Using the realization (30) of the γ-matrices,

$$\gamma^0 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ \gamma^j = \begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix},\ j = 1, 2, 3, \tag{48}$$

(where each entry stands for a 2×2 matrix) we obtain the system of equations

$$(\partial_0 + \boldsymbol{\sigma\partial})\psi_R + m\psi_L = 0 = (\boldsymbol{\sigma\partial} - \partial_0)\psi_L + m\psi_R. \tag{49}$$

(They split into two decoupled Weyl equations in the zero mass limit, the 4-dimensional counterpart of (45).)

We define the charge conjugate 4-component spinor ψ^C in accord with (41) by

$$\psi^C = \psi^*\gamma^0 C,\ C = \begin{pmatrix}c & 0\\ 0 & c\end{pmatrix} (\Rightarrow (\psi^C)^C = \psi). \tag{50}$$

One finds

$$B := \gamma^0 C = \begin{pmatrix}0 & c\\ -c & 0\end{pmatrix} = B^t,\ \psi_L^C = -\psi_R^* c,\ \psi_R^C = \psi_L^* c. \tag{51}$$

It was Majorana [38] who discovered (nine years after Dirac wrote his equation) that there exists a real representation, spanned by ψ_M of $Cl(3, 1)$, for which

$$\psi_M^C = \psi_M\ \Leftrightarrow \psi_R = \psi_L^* c\ (\psi_L = \psi_R^* c^{-1}). \tag{52}$$

[16] The first application of the Dirac equation dealt with the fine structure of the energy spectrum of hydrogen-like atoms that corresponds to a (central) Coulomb potential (see, e.g. [39], Sect. 14.1, as well as the text by Donkov and Mateev [**?**]). It was solved exactly in this case independently by Walter Gordon in Hamburg and by Charles Galton Darwin in Edinburgh, weeks after the appearance Dirac's paper (for a historical account see [40]). Our conventions for the Dirac equation, the chirality matrix γ_5 etc. coincide with Weinberg's text (see [39] Sect. 7.5) which also adopts the space-like Lorentz metric.

(Dirac equation was designed to describe the electron - a charged particle, different from its antiparticle. Majorana thought of applying his "real spinors" for the description of the neutrino, then only a hypothetical neutral particle - predicted in a letter by Pauli and named by Fermi[17].)

Sometimes, the Majorana representation is defined to be one with real γ-matrices. This is easy to realize (albeit not necessary) by just setting $\gamma_2^M = \gamma_5$ (which will give $\gamma_5^M := i\gamma_0^M\gamma_1^M\gamma_2^M\gamma_3^M = -\gamma_2$). In accord with Pauli lemma there is a similarity transformation (that belongs to $Spin(4,1) \subset Cl^0(4,1) \simeq Cl(4)$) between γ_μ^M and γ_μ (of Eq. (48)):

$$\gamma_\mu^M = S\gamma_\mu S^* \text{ for } S = \frac{1}{\sqrt{2}}(1 - \gamma_2\gamma_5)\,(S^* = \frac{1}{\sqrt{2}}(1 + \gamma_2\gamma_5)). \qquad (53)$$

The charge conjugation matrix C_M in the Majorana basis coincides with γ_0^M, the only skew-symmetric Majorana matrix while the symmetric form B_M of Eq. (41) is 1:

$$C_M = \gamma_0^M,\ B_M = \gamma_M^0 C_M = 1 \Rightarrow \psi^C = \psi^*. \qquad (54)$$

We prefer to work in the *chirality basis* (47) (called *Weyl basis* in [1]) in which γ_5 is diagonal (and the Lorentz /$Spin(3,1)$-/ transformations are reduced).

Exercise 4.1 Find the similarity transformation which relates the *Dirac basis* (with a diagonal γ_{Dir}^0),

$$i\gamma_{Dir}^0 = \gamma_5, \gamma_{Dir}^j = \gamma^j \ \Rightarrow C_{Dir} = i\gamma_2, \qquad (55)$$

to our chirality basis. Compute γ_{Dir}^5.

The Dirac quantum field ψ and its conjugate $\tilde{\psi}$, which describe the free electron and positron, are operator valued solutions of Eq. (48) that are expressed as follows in terms of their Fourier (momentum space) modes:

$$\begin{aligned} \psi(x) &= \textstyle\int (a_\zeta(p)e^{ipx}u_\zeta(p) + b_\zeta^*(p)e^{-ipx}v_\zeta(p))(dp)_m \\ \widetilde{\psi}(x)(= \psi^*(x)\beta) &= \textstyle\int (a_\zeta^*(p)e^{-ipx}\widetilde{u}_\zeta(p) + b_\zeta(p)e^{ipx}\widetilde{v}_\zeta(p))(dp)_m, \end{aligned} \qquad (56)$$

where summation in ζ (typically, a *spin projection*) is understood, spread over the two independent (classical) solutions of the linear homogeneous (algebraic) equations

$$\begin{aligned} &(m + ip\gamma)u_\zeta(p) = 0\Big(= \widetilde{u}_\zeta(p)(m + ip\gamma)\Big), \\ &(m - ip\gamma)v_\zeta(p) = 0, \text{ for } p^0 = \sqrt{m^2 + \boldsymbol{p}^2}, \end{aligned} \qquad (57)$$

[17] Enrico Fermi (1901-1954), Nobel Prize in Physics, 1938, for his work on induced radioactivity. It was he who coined the term *neutrino* – as a diminutive of neutron. (See E. Segrè, *Enrico Fermi – Physicist*, Univ. Chicago Press, 1970)

while $(dp)_m$ is the normalized Lorentz invariant volume element on the positive mass hyperboloid,

$$(dp)_m = (2\pi)^{-3}\frac{d^3p}{2p^0} = \Big(\int_0^\infty \delta(m^2+p^2)dp^0\Big)\frac{d^3p}{(2\pi)^3},\ p^2 = \boldsymbol{p}^2 - p_0^2. \quad (58)$$

The *creation* $(a^*_\zeta,\, b^*_\zeta)$ and the *annihilation* $(a_\zeta,\, b_\zeta)$ *operators* are assumed to satisfy the *covariant* canonical anticommutation relations

$$\begin{aligned}&[a_\zeta(p), a^*_{\zeta'}(p')]_+ = \delta_{\zeta\zeta'}(2\pi)^3 2p^0\delta(\mathbf{p}-\mathbf{p}') = [b_\zeta(p), b^*_{\zeta'}(p')]_+\\ &[a_\zeta(p), b^*_{\zeta'}(p')]_+ = 0 = [a_\zeta, b_{\zeta'}]_+ = \dots\end{aligned} \quad (59)$$

Stability of the ground state (or the *energy positivity*) requires that the vacuum vector $|0\rangle$ is annihilated by $a_\zeta,\, b_\zeta$

$$a_\zeta(p)|0\rangle = 0 = \langle 0|a^*_\zeta(p),\ b_\zeta(p)|0\rangle = 0 = \langle 0|b^*_\zeta(p). \quad (60)$$

This allows to compute the electron 2-point function

$$\langle 0|\psi(x_1)\otimes\widetilde{\psi}(x_2)|0\rangle = \int e^{ipx_{12}}(m - i\gamma p)(dp)_m,\ x_{12} = x_1 - x_2, \quad (61)$$

where we have fixed on the way the normalization of the solutions of Eq. (57),

$$\begin{aligned}&\sum_\zeta u_\zeta(p)\otimes\widetilde{u}_\zeta(p) = m - i\gamma p,\ \sum_\zeta v_\zeta(p)\otimes\widetilde{v}_\zeta(p) = -m - i\gamma p;\\ &\widetilde{u}_\eta(p)u_\zeta(p) = 2m\delta_{\eta\zeta} = -\widetilde{v}_\eta(p)v_\zeta(p).\end{aligned} \quad (62)$$

Remark 4.2 Instead of giving a basis of two independent solutions of Eq. (57) we provide *covariant* (in the sense of (27)) *pseudohermitean expressions* for the sesquilinear combinations (62). The idea of using bilinear characterizations of spinors is exploited systematically in [4].

Note that while the left hand side of (62) involves (implicitly) the matrix β, entering the *Dirac conjugation*

$$u \to \widetilde{u} = \bar{u}\beta\,(\psi \to \widetilde{\psi} = \psi^*\beta), \quad (63)$$

its right hand side is independent of β; thus Eq. (62) can serve to determine the phase factor in β. In particular, it tells us that β should be hermitean:

$$\begin{aligned}&(\sum_\zeta u_\zeta(p)\otimes\widetilde{u}_\zeta(p))^*\beta = \beta^*\sum_\zeta u_\zeta(p)\otimes\widetilde{u}_\zeta(p),\\ &(m - i\gamma p)^*\beta = \beta(m - i\gamma p)\ \Rightarrow \beta^* = \beta = \beta^\dagger.\end{aligned} \quad (64)$$

The positivity of matrices like $\sum_\zeta u_L(p,\zeta)\otimes\bar u_L(p,\zeta)$ for the chiral components of u (and similarly for v) - setting, in particular, $\widetilde{u} = i(-\bar u_R, \bar u_L)$ - fixes the remaining sign ambiguity (as $p^0 > 0$ according to (57)

$$\beta = i\gamma^0 \Rightarrow \sum_\zeta u_L(p,\zeta)\otimes\bar u_L(p,\zeta) = p^0 - \mathbf{p}\sigma =: \tilde p,$$
$$\sum_\zeta u_R(p,\zeta)\otimes\bar u_R(p,\zeta) = p^0 + \mathbf{p}\sigma = \underset{\sim}{p}\ (\underset{\sim}{p}\,\tilde p = -p^2 = m^2). \tag{65}$$

Using further the Dirac equation (57),

$$mu_L = i\tilde p u_R,\ mu_R = -i\underset{\sim}{p}u_L, \tag{66}$$

we also find

$$\sum_\zeta u_L(p,\zeta)\otimes\bar u_R(p,\zeta) = im = -\sum_\zeta u_R(p,\zeta)\otimes\bar u_L(p,\zeta). \tag{67}$$

Exercise 4.2 Deduce from (62) and from the definition (50) of charge conjugation that u^C can be identified with v; more precisely,

$$\sum_\zeta u^C_\zeta(p)\otimes\widetilde{u^C}_\zeta(p) = -m - i\gamma p (= \sum_\zeta v_\zeta(p)\otimes\widetilde{v}_\zeta(p)). \tag{68}$$

Any Dirac field can be split into a real and an imaginary part (with respect to charge conjugation):

$$\psi(x) = \frac{1}{\sqrt{2}}(\psi_M(x)+\psi_A(x)),\ \psi^C_M = \psi_M,\ \psi^C_A = -\psi_A,$$
$$\psi_M(x) = \int (c(p)u(p)e^{ipx} + c^*(p)u^C(p)e^{-ipx})(dp)_m,$$
$$\psi_A(x) = \int (d(p)u(p)e^{ipx} - d^*(p)u^C(p)e^{-ipx})(dp)_m. \tag{69}$$

The field ψ can then again be written in the form (56) with

$$\sqrt{2}a(p) = c(p) + d(p),\ \sqrt{2}b(p) = c(p) - d(p). \tag{70}$$

Remark 4.3 For a time-like signature, the counterpart of the Majorana representation for $Cl(1,3)$ would involve pure imaginary γ-matrices; the free Dirac equation then takes the form $(i\gamma\partial - m)\psi = 0$ (instead of (48). Such a choice seems rather awkward (to say the least) for studying real spinors.

5 Peculiarities of a Majorana Mass Term. Physical Implications

The Lagrangian density for the free Dirac field has the form

$$\mathcal{L}_0 = -\widetilde{\psi}(m + \gamma\partial)\psi. \tag{71}$$

(In the quantum case one should introduce normal ordering of the fields, but such a modification would not affect the conclusion of our formal discussion.) The mass term, $m\widetilde{\psi}\psi$ is non-vanishing for a Dirac field at both the quantum and the classical level (viewing, in the latter case, the components of ψ as /commuting/ complex-valued functions). This is, however, not the case for a Majorana field, satisfying (52). Indeed, the implication

$$\psi^C = \psi \Rightarrow \widetilde{\psi}\psi = i\psi C^{-1}\psi \tag{72}$$

of the reality of ψ tells us that the mass term vanishes for a classical Majorana field since the charge conjugation matrix is antisymmetric (in four dimensions). This is made manifest if we insert, using (52), the chiral components of the Majorana spinor:

$$\widetilde{\psi}\psi = i(\psi_L^*\psi_R - \psi_R^*\psi_L) = i(\psi_R c^{-1}\psi_R - \psi_L c\psi_L)\,(c = i\sigma_2 = -c^t). \tag{73}$$

Thus, the first peculiarity of a Majorana mass term is that it would be a purely quantum effect with no classical counterpart, in contrast to a naive understanding of the “correspondence principle”. An even more drastic departure from the conventional wisdom is displayed by the fact that the reality condition (52) (or, equivalently, (72)) is not invariant under phase transformation ($\psi \rightarrow e^{i\alpha}\psi$). Accordingly, the $U(1)$ current of an *anticommuting* Majorana field,

$$i\widetilde{\psi}\gamma^\mu\psi = \psi C\gamma^\mu\psi \tag{74}$$

vanishes since the matrix $C\gamma^\mu$ is symmetric as a consequence of the definition of C, (33). In particular, a Majorana neutrino coincides with its antiparticle implying a violation of the lepton number conservation, a consequence that may be detected in a neutrinoless double beta decay (see [36,41] and references therin) and may be also in a process of left-right symmetry restoration that can be probed at the Large Hadron Collider ([42,43]).

The discovery of neutrino oscillations is a strong indication of the existence of positive neutrino masses (for a recent review by a living classic of the theory and for further references - see [36]). The most popular theory of neutrino masses, involving a mixture of Majorana and Dirac neutrinos, is based on the so called *“seesaw mechanism”*, which we proceed to sketch (cf. [44] for a recent review with an eye towards applications to cosmological dark matter and containing a bibliography of 275 entries).

A model referred to as νMSM (for *minimal standard model with neutrino masses*) involves three Majorana neutrinos N_a $(a = 1, 2, 3)$ on top of the three known weakly interacting neutrinos, ν_α, that are part of three leptonic lefthanded doublets L_α $(\alpha = e, \mu, \tau)$. To underscore the fact that N_a are *sterile neutrinos* which do not take part in the standard electroweak interactions, we express them, using (52), in terms of *right handed* (2-component, Weyl) spinors R_a and their conjugate,

$$N_a = \begin{pmatrix} R_a^* c^{-1} \\ R_a \end{pmatrix}, \ a = 1, 2, 3. \tag{75}$$

The νMSM action density is obtained by adding to the standard model Lagrangian, $\mathcal{L}_{SM}$, an Yukawa interaction term involving the Higgs doublet H along with L_α and N_a and the free Lagrangian for the heavy Majorana fields

$$\mathcal{L} = \mathcal{L}_{SM} - \widetilde{N}_a(\gamma\partial + M_a)N_a - y_{\alpha a}(H^* \widetilde{L}_\alpha N_a + H\widetilde{N}_a L_\alpha) \tag{76}$$

with the assumption that

$$|y_{\alpha a}\langle H\rangle| \ll M_a, \tag{77}$$

where $\langle H\rangle$ is the vacuum expectation value of the Higgs field responsible for the spontaneous symmetry breaking that yields positive masses in the standard model.

In order to display the idea of the seesaw mechanism[18] we consider a two-by-two block of the six-by-six "mass matrix"

$$\begin{pmatrix} 0 & y\langle H\rangle \\ y\langle H\rangle & M \end{pmatrix}. \tag{78}$$

It has two eigenvalues M_N and $-m_\ell$, where under the assumption (77), $m_\ell <<$ M_N. Identifying m_ℓ with the light left neutrino mass, and M_N with the mass of the heavy sterile neutrino we find, approximately,

$$m_\ell \simeq \frac{(y < H >)^2}{M}, \ M_N \simeq M. \tag{79}$$

Historical note: *Bruno Pontecorvo*. Wolfgang Pauli (see footnote 12), who predicted the neutrino in a letter not destined for publication, did not believe that it could ever be observed. A physicist who did believe in the experimental study of the neutrinos was Bruno Pontecorvo[19] (1913-1993), aptly called Mr. Neutrino by his long-time (younger) collaborator Samoil M. Bilenky (see [46]). He

[18] This idea has been developed by a number of authors starting with P. Minkowski, 1977 - see for historical references [44] and for some new developments [**?**, 42]. Its counterpart in the noncommutative geometry approach to the standard model that uses the euclidean picture (in which there are no Majorana spinors) is discussed in [43] [45].

[19] The "Recollections and reflections about Bruno Pontecorvo" by S.S. Gershtein, available electornically in both the original Russian and in English, give some idea of this remarkable personality - which also emerges in Pontecorvo's own recollections [32].

proposed (in a 1946 report) a method for detecting (anti)neutrino in nuclear reactors, a methodology used by Frederick Reines (1918-1999) and Clyde Cowan (1919-1974) in their 1956 experiment that led to the discovery of neutrino (for which the then nearly 80-year-old Reines shared the Nobel Prize in Physics in 1995). Pontecorvo predicted that the muon neutrino may be different from the electron one and proposed an experimental method to prove that in 1959. His method was successfully applied three years later in the Brookhaven experiment for which J. Steinberger, L. Lederman and M. Schwarz were awarded the Nobel Prize in 1988. He came to the idea of neutrino oscillation in 1957 and from then on this was his favourite subject. Vladimir Gribov (1930-1997) and Pontecorvo considered in 1969 the possibility of lepton number violation through a Majorana mass term and applied their theory to the solar neutrino problem. Bilenky and Pontecorvo introduced the general Majorana-Dirac mass term that is used in the seesaw mechanism [47]. (See for details and references [46].) Neutrino oscillations are now well established in a number of experiments - and await another Nobel Prize triggered by the formidable intuition of Bruno Pontecorvo.

Acknowledgments

Clifford algebras have fascinated mathematical physicists all over the world. I have benefited, in particular, from conversations with Petko Nikolov and Ludwik Dabrowski who have popularized them at the University of Sofia and in Italy, respectively (see [48] and [19]). I also thank Samoil Mihelevich Bilenky and Serguey Petcov for teaching me the physics of Majorana neutrinos.

The author thanks the High Energy Division of The Abdus Salam International Centre for Theoretical Physics (ICTP) and the Theory Group of the Physics Department of CERN, where these notes were completed. Partial support by grant DO 02-257 of the Bulgarian National Science Foundation is gratefully acknowledged.

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