Dispersion Relation for MHD Waves – Reinvestigated

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Abstract. Kumar et al. [1] derived a fifth degree polynomial for the dispersion relation in \( \omega \) and pointed out that the calculations of Porter et al. [2] and of Dwivedi & Pandey [3] seemed to be in error, as they obtained a sixth degree polynomial. Dwivedi & Pandey [4] not only protected their sixth degree polynomial, but attacked the fifth degree polynomial. Both groups have used the same set of basic equations. In another paper, Pandey & Dwivedi [5] have again tried to show that the dispersion relation must be a sixth degree polynomial. In the present communication, we have derived a seventh degree polynomial for the dispersion relation, using the same set of basic equations. Further, we have shown that the results for the fast and slow modes of magnetohydrostatic waves are independent of the degree of polynomial for the dispersion relation and the fifth degree polynomial is sufficient.

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1 Introduction

For application of magnetohydrodynamics (MHD) in solar physics as well as in plasma physics, dispersion relation plays key role. The basic equations under the present investigation can be expressed as [1,5]

\[
\frac{\partial p}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \\
\rho \frac{\partial \vec{v}}{\partial t} = -\nabla p + \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} - \nabla \cdot \Pi, \\
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}), \\
\frac{\partial p}{\partial t} + \gamma p(\nabla \cdot \vec{v}) = (\gamma - 1)[\nabla \cdot \kappa \nabla T + Q_{\text{vis}} - Q_{\text{rad}}], \\
p = \frac{2pB^2}{m_p}.
\]

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Figure 1. The applied magnetic field is along $z$-axis whereas the propagation vector $\vec{k}$ lies in the $z$-$x$ plane.

Here, symbols have their usual meaning. The quantities $Q_{\text{th}}$, $Q_{\text{vis}}$ and $Q_{\text{rad}}$ are

\[ Q_{\text{th}} = k_\parallel \left( \frac{\partial T}{\partial z} \right)^2 T^{-1}, \]
\[ Q_{\text{vis}} = \frac{\eta_0}{3} (\nabla \cdot \vec{v})^2, \]
\[ Q_{\text{rad}} = n_e n_H Q(T), \]

where $\kappa_\parallel$ represents the conductivity along the magnetic field and is expressed by $\kappa_\parallel \approx 10^{-6} T^{5/2}$. $Q_{\text{vis}}$ is the volumetric heating rate due to viscosity; $Q_{\text{th}}$ is the volumetric heating rate due to electron thermal conduction; $Q_{\text{rad}}$ stands for the radiative loss rate per unit volume. More precisely, $Q_{\text{vis}} = -\Pi_{\alpha,\beta} \left( \frac{\partial v_\alpha}{\partial x_\beta} \right)$ expressed by Braginskii [6]. In equation (3), the term $\eta \nabla^2 \vec{B}$ is not accounted for, as the value of the magnetic Raynold number is quite large ($R_m \approx 10^9$) for the region considered. For small perturbations from the equilibrium, we have

\[ \rho = \rho_0 + \rho_1, \quad \vec{v} = \vec{v}_1, \quad \vec{B} = \vec{B}_0 + \vec{B}_1, \]
\[ \rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\nabla p_0 + \frac{1}{4\pi} \nabla \times (\nabla \times \vec{B}_1) \times \vec{B}_0 - \nabla \cdot \Pi_0, \]
\[ \frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v}_1 \times \vec{B}_0), \]

where the equilibrium part is denoted by the subscript ‘0’ and the perturbation part by the subscript ‘1’. For the magnetic field taken along the $z$-axis, (i.e., $\vec{B}_0 = B_0 \hat{z}$) and the propagation vector $\vec{k} = k_\perp \hat{x} + k_\parallel \hat{z}$, as shown in Figure 1.

The equations (1)–(5) can be linearized in the following form:

\[ \frac{\partial \rho_0}{\partial t} + \rho_0 (\nabla \cdot \vec{v}_1) = 0, \quad (6) \]
\[ \rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\nabla p_0 + \frac{1}{4\pi} \nabla \times (\nabla \times \vec{B}_1) \times \vec{B}_0 - \nabla \cdot \Pi_0, \quad (7) \]
\[ \frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v}_1 \times \vec{B}_0), \quad (8) \]
For the perturbations that are proportional to $\exp[i(\vec{k} \cdot \vec{r} - \omega t)]$, equations (6)–(10) reduce to the following equations:

\[
\begin{align*}
\omega p_1 - \rho_0(k_x v_{1x} + k_z v_{1z}) &= 0, \quad (11) \\
\omega p_0 v_{1x} - k_x p_1 &= \frac{B_0}{4\pi}(k_x B_{1z} - k_z B_{1x}) + \frac{i\rho_0}{3} (k_x^2 v_{1x} - 2k_x k_z v_{1z}) = 0, \quad (12) \\
\omega p_0 v_{1y} + \frac{B_0}{4\pi}(k_z B_{1y}) &= 0, \quad (13) \\
\omega p_0 v_{1z} - k_z p_1 + \frac{i\rho_0}{3} (4k_z^2 v_{1z} - 2k_x k_z v_{1x}) &= 0, \quad (14) \\
\omega B_{1x} + k_z B_0 v_{1z} &= 0, \quad (15) \\
\omega B_{1y} + k_z B_0 v_{1y} &= 0, \quad (16) \\
\omega B_{1z} - k_x B_0 v_{1x} &= 0, \quad (17) \\
i\omega p_1 - i\rho_0 c_A^2 (k_x v_{1x} + k_z v_{1z}) - (\gamma - 1) \kappa || k_z^2 T_1 &= 0, \quad (18) \\
\frac{p_1}{p_0} - \frac{p_1}{p_0} - \frac{T_1}{T_0} &= 0. \quad (19)
\end{align*}
\]

Equations (13) and (16) for the variables $v_{1y}$ and $B_{1y}$ are decoupled from the rest and describe Alfvén waves. The rest of the equations for $p_1$, $p_0$, $T_1$, $B_{1x}$, $B_{1z}$, $v_{1x}$ and $v_{1z}$ describe damped magnetohydrostatic waves. Now, on substituting $B_{1x}$ and $B_{1z}$ from equations (15) and (17) in equations (12) and (14), we get

\[
\left(\omega^2 \rho_0 + \frac{i\omega_\eta k_z k_x^2 - \nu_A^2 \rho_0 k_z^2}{3}\right) v_{1x} - \frac{2i\omega_\eta k_z k_x}{3} v_{1z} - k_x \omega p_1 = 0 \quad (20)
\]

and

\[
\frac{2i\omega_\eta k_z k_x}{3} v_{1z} - \left(\omega^2 \rho_0 + \frac{4i\omega_\eta k_z}{3}\right) v_{1z} + k_z p_1 = 0. \quad (21)
\]

When we eliminate $p_1$ and $T_1$ from equations (11), (18) and (19), we get

\[
(c_0 \rho_0 k_x - i\rho_0 c_A^2 k_z \omega) v_{1x} + (c_0 \rho_0 k_z - i\rho_0 c_A^2 k_z \omega) v_{1z} - (c_0 \omega - i\omega^2) p_1 = 0, \quad (22)
\]

where $c_0 = (\gamma - 1) \kappa || T_0 / \rho_0$, $c_A^2 = \gamma p_0 / \rho_0$ and $v_A^2 = B_0^2 / 4\pi \rho_0$. This is the juncture where the conflict arises. Now, our aim is to derive dispersion relations.

## 2 Dispersion Relations

The dispersion relations obtained from equations depend on the procedure for solving this set of equations. For convenience, let us express equations (20)–
(22) as

\[ a_{11}v_{1x} + a_{12}v_{1z} + a_{13}p_1 = 0, \]  
\[ a_{21}v_{1x} + a_{22}v_{1z} + a_{23}p_1 = 0, \]  
\[ a_{31}v_{1x} + a_{32}v_{1z} + a_{33}p_1 = 0, \]  

where the coefficients \( a_{ij} \) are

\[ a_{11} = \left( \omega^2 \rho_0 + \frac{i\omega \eta_0 k_x^2}{3} - v_A^2 \rho_0 k_x^2 \right), \]
\[ a_{12} = -\frac{2i\omega \eta_0}{3} k_x k_z, \]
\[ a_{13} = -k_x \omega, \]
\[ a_{21} = \frac{2i\eta_0}{3} k_x k_z, \]
\[ a_{22} = -\omega \rho_0 - \frac{4i\eta_0}{3} k_x^2, \]
\[ a_{23} = k_z, \]
\[ a_{31} = c_0 \rho_0 k_x - i \rho_0 c_s^2 k_z \omega, \]
\[ a_{32} = c_0 \rho_0 k_z - i \rho_0 c_s^2 k_z \omega, \]
\[ a_{13} = i\omega^2 - c_0 \omega. \]

On eliminating \( v_{1x} \) from equations (23) and (24), we get

\[ (a_{12}a_{21} - a_{22}a_{11})v_{1z} + (a_{13}a_{21} - a_{23}a_{11})p_1 = 0. \]  

On eliminating \( v_{1z} \) from equations (24) and (25), we get

\[ (a_{22}a_{31} - a_{32}a_{21})v_{1z} + (a_{23}a_{31} - a_{33}a_{21})p_1 = 0. \]  

On eliminating \( v_{1z} \) from equations (23) and (25), we get

\[ (a_{12}a_{31} - a_{32}a_{11})v_{1z} + (a_{13}a_{31} - a_{33}a_{11})p_1 = 0. \]  

With the help of equations (26) and (27), we get

\[ \omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE = 0, \]  

where

\[ A = c_0 + \frac{\eta_0}{3\rho_0} (k_x^2 + 4k_z^2), \]
\[ B = \frac{c_0 \eta_0}{3\rho_0} (k_x^2 + 4k_z^2) + (c_s^2 + v_A^2) k_x^2, \]
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\[ C = \frac{3\eta_0}{\rho_0} c_s^2 k_x^2 k_z^2 + \frac{c_0 p_0 k^2}{\rho_0} + v_A^2 c_0 k^2 + \frac{4\eta_0 v_A^2 k_x^2 k_z^2}{3\rho_0}, \]

\[ D = \frac{3c_0 p_0 \eta_0 k_x^2 k_z^2}{\rho_0^2} + \frac{4\eta_0 c_0 v_A^2 k_x^2 k_z^2}{3\rho_0} + v_A^2 c_0^2 k_x^2 k_z^2, \]

\[ E = \frac{v_A^2 c_0 p_0 k_x^2 k_z^2}{\rho_0}. \]

This dispersion relation is the same as obtained by Kumar et al. [1]. With the help of equations (26) and (28), we get

\[ \left( \omega^2 + \frac{i\eta_0 k_x^2}{3\rho_0} \omega - v_A^2 k_x^2 \right) \left( \omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE \right) = 0. \quad (30) \]

It is a seventh degree polynomial in \( \omega \), which has been derived the first time. So far fifth and sixth degree polynomials are derived. Now, with the help of equations (27) and (28), we get

\[ \left( \omega + \frac{ic_0}{\gamma} \right) \left( \omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE \right) = 0. \quad (31) \]

This is a sixth degree polynomial. Thus, we obtained three polynomials of fifth, sixth and seventh degrees.

3 Discussion

This sixth degree polynomial is different from that derived by Dwivedi & Pandey [4] and Pandey & Dwivedi [5]. Notice that for solving the equations (23)–(25), we first eliminated \( v_1x \). On the other side, when we first eliminate \( p_1 \) from equations (23)–(25), we get the fifth degree polynomial (29) and a sixth degree polynomial

\[ (\omega + ic_0) \left( \omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE \right) = 0. \quad (32) \]

This is the sixth degree polynomial derived by Dwivedi & Pandey [4] and Pandey & Dwivedi [5].

It is interesting to note that in all the polynomials, the equation (29) is common. Hence five roots of all the polynomials are common. Other roots (sixth root for the sixth degree polynomial, and sixth and seventh roots for the seventh degree polynomial) depend on the method used in solving the equations (23)–(25). For the fast and slow modes of magnetohydrostatic waves, we require the complex roots only. These required complex roots are the roots of equation (29). Other roots are pure imaginary. Hence, the degree of polynomial does not affect the results for the fast and slow modes of magnetohydrostatic waves. The roots of equation (29) are as follows: \(-i\alpha_{1r}, \pm i\alpha_{2r}, \pm i\alpha_{3r}\), satisfying the relations
\(\alpha_1 r + 2\alpha_2 r + 2\alpha_3 r = A,\)
\(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_1\alpha_2 + 2\alpha_1\alpha_3 + 4\alpha_2\alpha_3 = B,\)
\(\alpha_1\alpha_2^2 + \alpha_1\alpha_3^2 + 4\alpha_1\alpha_2\alpha_3 + 2\alpha_2^2\alpha_3 + 2\alpha_2\alpha_3^2,\)
\(2\alpha_2^2\alpha_1\alpha_3 + \alpha_2^2\alpha_1\alpha_3 + \alpha_2^2\alpha_1\alpha_3 + \alpha_2^2\alpha_3 = C,\)
\(\alpha_1\alpha_2^2\alpha_3^2 + \alpha_1\alpha_2\alpha_3^2 + \alpha_1\alpha_2\alpha_3^2 + \alpha_1\alpha_2\alpha_3^2 = E.\)

Other roots (one for the sixth degree polynomial and two for the seventh degree polynomial) have no effect on the fast and slow modes of magnetohydrostatic waves. Moreover, those roots are not the actual ones. Since the equations (23)–(25) are homogeneous, the proper way to solve them is to evaluate the determinant
\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = 0.
\]

Solution of this determinant gives equation (29). Hence, the actual dispersion relation is the fifth degree polynomial. Though we can get sixth and seventh degree polynomials, but there is no effect on the fast and slow modes of magnetohydrostatic waves.

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