

Conformal Quantum Field Theory in Various Dimensions

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Abstract. Various relations between conformal quantum field theories in one, two and four dimensions are explored. The intention is to obtain a better understanding of 4D CFT with the help of methods from lower dimensional CFT.

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1 Introduction

Quantum field theory (QFT) in four spacetime dimensions (4D) continues to be a great challenge after many decades of intense research. While perturbative and nonperturbative approximation schemes have proven most efficient for many purposes, the rigorous construction of nontrivial theories has not been achieved.

On the other hand, in two dimensions (2D), many nontrivial models have been constructed. A huge body of model-independent knowledge has been accumulated in particular for conformal field theories; depending on the value of the central charge, there are even classifications available.

To close the gap between 2 and 4 dimensions, one would like to be able to transfer general knowledge from 2 to 4. This is (besides its manifold statistical mechanic applications) the main *raison d'être* for the study of low-dimensional models. In this contribution, we present a number of attempts hopefully leading to new insights into the structure of correlation functions in nontrivial 4D conformal QFT which is admissible from an axiomatic point of view.

To have the maximum power of this approach available, we assume the strongest form of conformal symmetry, called “global conformal invariance” (GCI) in [1,

2]: the conformal group is implemented by a *true* representation on the Hilbert space. This implies that the covariant fields have integer scaling dimensions and satisfy Huygens' principle, *i.e.*, they commute not only at spacelike but also at timelike distance. Moreover, their correlation functions are rational, and in fact polynomial after multiplication with sufficiently high powers of Lorentz square distances $\rho_{ij} = (x_i - x_j)^2$. While these features are conspicuously close to free field theory, we shall indicate below why we expect nontrivial fields within this highly restricted class. Notice that the massless free field in $D > 2$ dimensions has scaling dimension $(D - 2)/2$, so this field does not satisfy GCI if D is odd. Recall also that in $D = 2$, the massless free field does not exist because it is too singular at zero momentum, but its gradient $j_\mu = \partial_\mu \varphi$ can be defined. It is a conserved vector current of scaling dimension 1 and decomposes into two chiral fields $j_0(x) \pm j_1(x) = j_\pm(x^0 \pm x^1)$.

In $D \geq 4$ even spacetime dimensions, GCI proved to be a highly restrictive symmetry. In Section 2, we discuss the leeway it allows beyond free fields in terms of the pole structure of correlation functions. Remarkably, the new features can arise only in at least six-point correlations – which are hardly ever studied!

The main open question is, whether this leeway is compatible with Hilbert space positivity. A powerful method to approach this question for four-point correlations is the partial wave expansion; unfortunately, the partial waves are not known for more than four points. Part of the subsequent sections about “restriction” is motivated by attempts to find alternative approaches to positivity applicable to higher correlation functions, to which we return in Section 5.

2 Conservation Laws

2.1 Conserved Tensor Fields

Consider conformal symmetric traceless tensor fields of rank r and scaling dimension d . The quantity $d - r$ is called “twist”. The fields of twist $D - 2$, where D is the spacetime dimension, are distinguished: their two-point function is determined by conformal invariance and turns out to have zero divergence. By the Reeh-Schlieder theorem it follows that these fields are conserved tensor fields:

$$\partial_\mu T^{\mu \dots \nu} = 0, \tag{1}$$

with the exception of $r = 0, d = D - 2$. Except for the scalars, the twist $D - 2$ fields have the lowest possible dimension for the given tensor rank admitted by the unitarity bound [3]. The scalars have been proven, in $D = 4$ dimensions [4], to be either Wick squares of massless free fields, or generalized free fields. We expect that a similar argument holds also in $D = 2n = 6, 8, \dots$ dimensions for scalar fields of scaling dimension $D - 2$. (This cannot be true for odd D because the massless free field violates GCI, and also not for $D = 2$ because the massless free field does not exist.)

In $D = 2$, the distinguished fields are precisely those which decompose into chiral fields: Symmetric traceless tensors have only two independent components T_{++++} and T_{----} , and by the conservation law, these depend only on $x^0 \pm x^1$. Indeed, almost all our knowledge about 2D CFT relies on the presence of these distinguished chiral fields, such as currents or the stress-energy tensor.

2.2 Biharmonic Fields

Also in even dimension $D \geq 4$, the presence of conserved tensor fields has far reaching consequences. For definiteness, $D = 4$ throughout this section, although the statements generalize to even $D \geq 4$. The operator product expansion of any pair of fields A and B can be organized according to the twist of the composite fields. If A and A' are scalar of equal dimension d , then the lowest contribution to $A'(x)A(y)$ after the vacuum contribution is that of twist $D - 2$. Multiplying this contribution by $((x - y)^2)^{d-1}$, one arrives at a “bifield” $V_{A'A}(x, y)$, while all higher twist contributions are of higher order in $(x - y)^2$. The infinitely many conservation laws for the local fields comprised in $V_{A'A}$ can be cast into the simple form, called “biharmonicity” [5]

$$\square_x V_{A'A}(x, y) = 0 = \square_y V_{A'A}(x, y). \quad (2)$$

Biharmonicity is a highly nontrivial feature. By a classical result [6], every power series p in $z \in \mathbb{R}^n$ has a unique “harmonic completion” $h = p + z^2 \cdot q$, such that h is harmonic: $\square_z h = 0$, and q is another power series. But correlation functions involving $V_{A'A}(x, y)$ are harmonic both w.r.t. x and w.r.t. y . Therefore, the contribution from $V_{A'A}(x, y)$ in a correlation functions involving $A'(x)A(y)$ must coincide with two *a priori* different harmonic completions (w.r.t. $z = x - y$). The condition that the two completions coincide is found, for purely scalar correlations, to be a universal third order linear partial differential equation to be satisfied by the function U_0 defined by

$$\langle \cdots [A'(x)A(y) - \langle A'(x)A(y) \rangle] \cdots \rangle = \frac{1}{((x - y)^2)^{d-1}} (U_0 + O((x - y)^2)),$$

where U_0 is a Laurent polynomial in the Lorentz square distances $\rho_{xi} = (x - x_i)^2$, $\rho_{yi} = (y - x_i)^2$, and $\rho_{ij} = \rho_{ji} = (x_i - x_j)^2$ (x_i are the coordinates of the other scalar fields in the correlation), homogeneous of degree -1 in both sets of variables ρ_{xi} and ρ_{yi} . When this condition is fulfilled, the contribution from $V_{A'A}$ to the above correlation is the unique biharmonic completion V_0 of U_0

$$\langle \cdots V_{A'A}(x, y) \cdots \rangle = U_0 + O((x - y)^2).$$

The PDE to be satisfied by U_0 reads

$$\left[\left(\sum_i \rho_{yi} \partial_{\rho_{xi}} \right) \left(\sum_{i < j} \rho_{ij} \partial_{\rho_{yi}} \partial_{\rho_{yj}} \right) - (x \leftrightarrow y) \right] U_0 = 0. \quad (3)$$

Together with rationality, it is highly restrictive and constrains the admissible form of U_0 far beyond conformal invariance. In [4], it was shown that the only poles of U_0 in the arguments $x, y \in \mathbb{R}^4$ can be of the form

$$\frac{P}{\rho_{xm}^a \rho_{xn}^b \cdot \rho_{ym}^c \rho_{yn}^d} \quad \text{with } a, b, c, d \geq 0,$$

for some pair $m \neq n$, where P is polynomial in ρ_{xi} and ρ_{yi} ($i \neq m, n$), and a Laurent polynomial in all other squared Lorentz distances ρ_{ij} . We call this structure a “double pole” if both a and b are positive, or if both $c, d > 0$.

The relevance of this observation is the following: Free field examples of biharmonic fields are $:\varphi(x)\varphi(y):$ and $:\bar{\psi}(x)(x_\mu - y_\mu)\gamma^\mu\psi(y):$, where φ and ψ are the free massless scalar and Dirac field. But correlation functions of Wick products of free fields and their derivatives can only produce “single poles” with $a = 0$ or $b = 0$, and $c = 0$ or $d = 0$. Therefore, any double pole is a clear signal of a nontrivial GCI CFT. On the other hand, double poles cannot arise in four-point functions just because there are not sufficiently many variables. Therefore, this signal can only be seen in at least five-point correlations [4].

An example of a six-point double pole structure was presented in [4]. A more systematic study was made by one of us [7]. For a double pole as above, we call $a + b + c + d$ its order. A double pole structure (DPS) is a rational solution to the PDE (3)

$$\sum_{a,b,c,d} \frac{P_{abcd}}{\rho_{xm}^a \rho_{xn}^b \cdot \rho_{ym}^c \rho_{yn}^d}$$

involving nonzero terms with a and $b > 0$, or c and $d > 0$. Their polynomial (in ρ_{xi} and ρ_{yi} , $i \neq m, n$) coefficient functions turn out to be organized into multiplets of $sl(2)$, whose generators are the differential operators

$$2H = \sum_{i \neq m, n} \rho_{xi} \partial_{\rho_{xi}} - \rho_{yi} \partial_{\rho_{yi}}, \quad X = \sum_{i \neq m, n} \rho_{xi} \partial_{\rho_{yi}}, \quad Y = \sum_{i \neq m, n} \rho_{yi} \partial_{\rho_{xi}}.$$

More precisely, every DPS is a linear combination of DPSs obtained as follows. Fix a pair of indices m, n . Fix four integers $0 \leq p < a$, $0 \leq q < b$. Let $\ell = p + q$ and choose a monomial P_ℓ of order ℓ in the variables ρ_{xi} ($i \neq m, n$). P_ℓ is then a highest weight vector of $sl(2)$: $HP_\ell = \frac{\ell}{2}P_\ell$ and $XP_\ell = 0$. Let $k = a + b - \ell - 1 \geq 1$ and choose a monomial Q_k of order k in the $sl(2)$ singlet variables $R_{ij} = \rho_{xi}\rho_{yj} - \rho_{xj}\rho_{yi}$ ($i, j \neq m, n$). Notice that for five-point correlations, such singlets are not available, hence one can also exclude five-point DPSs. These data, together with a Laurent monomial L in the variables ρ_{ki} so as to saturate the scaling dimension of the scalar fields in the correlation function, induce DPSs of maximal order $\mu = 2(a + b) - \ell$, whose double poles of order $= \mu$ are given in closed form by

$$\sum_{\delta=0}^p \sum_{\varepsilon=0}^q \frac{\rho_{xm}^\delta \rho_{xn}^\varepsilon \cdot \rho_{ym}^{q-\delta} \rho_{yn}^{p-\varepsilon}}{\rho_{xm}^a \rho_{xn}^b \cdot \rho_{ym}^c \rho_{yn}^d} \frac{(b-q)_\delta (-p)_\delta}{(1-a)_\delta \delta!} \frac{(a-p)_\varepsilon (-q)_\varepsilon}{(1-b)_\varepsilon \varepsilon!} \left| \frac{\ell}{2}, \frac{\ell}{2} - \delta - \varepsilon \right\rangle \cdot Q_k \cdot L,$$

where the functions $|\frac{\ell}{2}, \frac{\ell}{2} - \nu\rangle = (-1)^\nu (-\ell)_\nu Y^\nu P_\ell$ are vectors of weight $\frac{\ell}{2} - \nu$ in the irreducible highest weight $sl(2)$ module generated by $P_\ell = |\frac{\ell}{2}, \frac{\ell}{2}\rangle$. These contributions exhaust a two-dimensional rectangular sublattice within the lattice $a + b + c + d = \mu$. The poles of order $< \mu$ are then determined recursively from those of maximal order $= \mu$, because equation (3) connects different orders. The system is in fact overdetermined, but in all cases studied it could be solved. We conjecture that this is always the case. The solution is unique up to DPSs of lower maximal order.

Once the solution U_0 to (3) is given, its biharmonic completion, *i.e.*, the corresponding correlation function $\langle \cdots V_{A'A}(x, y) \cdots \rangle$ solving (2), can be computed recursively as a power series in $(x - y)^2$. Unlike the correlations of local fields, these correlations are always transcendental functions if U_0 contains double poles. In this case, $V_{A'A}$ cannot be Huygens bilocal, but is presumably Einstein bilocal in general, as a case study in [8] indicates.

3 Restrictions

3.1 Timelike Surfaces

The restriction of a quantum field to a timelike hypersurface yields another Wightman field in lower dimensions [9]. In this way, 4D fields give rise to 3D and to 2D fields. It is also known that conformal fields restrict to conformal fields on the hypersurface, and the decomposition of conformal tensor fields can be described in terms of “internal derivatives” of the original fields [6, 10, 11].

One can ultimately restrict the field to the time axis: because of Huygens locality, this yields a local conformal 1D field depending only on x^0 . Notice that this step is quite different from the decomposition of conserved 2D tensor fields into their chiral components, that depend only on $x^0 \pm x^1$. Yet, in both cases one arrives at Möbius covariant chiral fields!

To give an example: The correlation functions of restricted fields are just the restrictions of the original correlation functions. In particular, free fields remain free in the sense that the truncated correlations remain zero. Thus, if we restrict the massless free field φ in $D = 4$ to the plane $x^2 = x^3 = 0$, we arrive at a generalized free field with the two-point function

$$D(x - y)|_{\mathbb{R}^2} = \frac{(2\pi)^{-2}}{(x^1 - y^1)^2 - (x^0 - y^0 - i\varepsilon)^2}$$

of scaling dimension $d = 1$. But because the spacetime dimension has changed, its Källén-Lehmann weight is no longer a δ -function at $m^2 = 0$ but a continuum of all masses integrated with the measure dm^2 . Such fields do not possess a stress-energy tensor as a Wightman field, because its two-point function diverges [12]. Formally, one may assign an “infinite central charge” to this SET. One may

actually represent the generalized free field in 2 dimensions as a “central limit” $n \rightarrow \infty$ of

$$\varphi_n(x) = n^{-\frac{1}{2}} \sum_{\nu=1}^n \psi_\nu(x^0 + x^1) \otimes \psi_\nu(x^0 - x^1)$$

where ψ_ν are n independent chiral real free Fermi fields, hence the SET for ϕ_n has central charge $c = \frac{n}{2} \rightarrow \infty$.

On the other hand, restricting φ to the time axis, its two-point function is just

$$D(x-y)|_{\mathbb{R}} = (2\pi)^{-2} \left(\frac{-i}{x^0 - y^0 - i\varepsilon} \right)^2,$$

the two-point function of a canonical chiral current $j(x^0)$. The Wick square $:\varphi^2(x):$ restricts to $:j^2(x^0): = \pi^{-1}T(x^0)$, where T is the chiral stress-energy tensor with $c = 1$.

3.2 Null Surfaces

A different option is the restriction to null hypersurfaces such as $N = \{x \in \mathbb{R}^4 : x^0 = x^1\}$. This case is not covered by the result in [9]. Yet, *massive* free scalar fields can be restricted. More precisely, the naive restriction has an infrared singularity, which can be cured by taking a derivative w.r.t. x_+ , where $x_\pm = x^0 \pm x^1$. Then, defining $\varphi_N(x_+, x_\perp) := \partial_+ \varphi_m(x)|_{x_- = 0}$, one computes

$$\langle \varphi_N(x_+, x_\perp) \varphi_N(y_+, y_\perp) \rangle = \frac{1}{4\pi} \delta(x_\perp - y_\perp) \cdot \left(\frac{-i}{x_+ - y_+ - i\varepsilon} \right)^2. \quad (4)$$

This restriction is an instance of the more general situation studied in [13]. The result is nothing but an infinite system of canonical free currents $j_n(x_+) = \int d^2x_\perp \varphi_N(x_+, x_\perp) f_n(x_\perp)$, where f_n is an orthonormal basis of $L^2(\mathbb{R}^2)$. The remarkable fact is that the vacuum fluctuations associated with the transverse coordinates $x_\perp \in \mathbb{R}^2$ are completely suppressed [14], and these degrees of freedom are traded into an infinite-dimensional inner symmetry. Moreover, the restriction is *independent of the original mass*.

Looking at the field as a distribution, the construction means that the extension to test functions of the form $f(x_+, x_\perp) \delta(x_-)$ must be bought by the constraint that $f = \partial_+ g$, where g is a test function on \mathbb{R}^3 . Because the restriction is independent of the mass, *every* scalar two-point function restricts to the same result (4) times the integral over the Källen-Lehmann density. In particular, two-point functions of scalar fields where this integral is divergent cannot be restricted in the same way, such as the Wick square or non-superrenormalizable interacting fields. Moreover, the derivatives ∂_+ do not properly cure at the same time the single contraction terms appearing in higher correlation functions.

However, one can restrict the bifield $:\varphi_m(x)\varphi_m(y):$ via

$$\partial_{x_+}\partial_{y_+}:\varphi(x)\varphi(y):|_{x_-=y_-=0} = :\varphi_N(x_+, x_\perp)\varphi_N(y_+, y_\perp):.$$

(One may then well pass to coinciding points $x_+ = y_+$ after taking the derivatives and smearing in the transversal space \mathbb{R}^2 , but this is obviously not an operation on the Wick square itself.)

For $m = 0$, the bifield $:\varphi(x)\varphi(y):$ is the simplest instance of a biharmonic field, as discussed in the previous section. This suggests a speculation that biharmonic fields can always be restricted. This expectation is supported by the solution to the characteristic initial value problem for the wave operator in 4 dimensions, see (5) below with $m = 0$, which immediately generalizes to bifields. We leave this here as a conjecture, as another remarkable feature related to the distinguished twist $D - 2$ fields and their conservation laws.

3.3 An exotic Restriction?

The action of the group $SO(2, D)$ on the null cone $\xi \cdot \xi = (\xi^0)^2 - (\xi^1)^2 - \dots - (\xi^D)^2 + (\xi^{D+1})^2 = 0$ in $D + 2$ dimensions induces an action of $SO(2, D)/\mathbb{Z}_2$ on the projective cone obtained by the identification $\xi \sim \lambda\xi$ ($\lambda \in \mathbb{R} \setminus \{0\}$). The projective cone is known as the Dirac space or conformally compactified Minkowski spacetime $\overline{M}_D \sim (S^1 \times S^{D-1})/\mathbb{Z}_2$, into which D -dimensional Minkowski spacetime is embedded as the chart

$$x^\mu = \frac{\xi^\mu}{\xi^D + \xi^{D+1}} \quad (\mu = 0, \dots, D - 1),$$

so that $SO(2, D)/\mathbb{Z}_2$ becomes the conformal group. Restricting a 4D conformal QFT to 2D, the relevant conformal group is $SO(2, 2)/\mathbb{Z}_2 \subset SO(2, 4)/\mathbb{Z}_2$, embedded as the subgroup that fixes the restricted directions 2 and 3. This 2D conformal group is a direct product of two Möbius groups $SO(1, 2) = SL(2, \mathbb{R})/\mathbb{Z}_2 = SU(1, 1)/\mathbb{Z}_2$ acting on the chiral variables $x^0 \pm x^1$.

There is another embedding of two commuting Möbius subgroups $SO(1, 2)$ into $SO(2, 4)/\mathbb{Z}_2$ as the subgroups that fix the directions 0, 1, 2 and 3, 4, 5 respectively. One might wonder whether this subgroup G corresponds to some “exotic” 2D restriction.

The first objection is that G has no two-dimensional orbits in the 4D Dirac space \overline{M}_4 , that could serve as the restricted 2D world hypersurface. But one could envisage a more abstract situation following an idea of [15]: Let $\alpha_g^{(2)}$ denote the action of G on the 2D Dirac space \overline{M}_2 , and fix any double cone $O \subset \overline{M}_2$. Suppose we find a subalgebra A on the Hilbert space of the unrestricted 4D theory (where G is unitarily represented) with the properties that $U(g)AU(g)^* \subset A$ for all $g \in G$ such that $\alpha_g^{(2)}O \subset O$, and $U(g')AU(g')^*$ commutes with A for all

$g' \in G$ such that $\alpha_{g'}^{(2)}O \subset O'$, where O' is the causal complement of O in \overline{M}_2 . In this case, we may consistently *define*

$$A(\alpha_g^{(2)}O) := U(g)AU(g)^*$$

for all $g \in G$. These algebras on the Hilbert space of the 4D theory would then qualify as local algebras of a 2D CFT, satisfying local commutativity, conformal covariance and isotony. The problem with this is, however, that the L_0^\pm generators of the embedded subgroup do not have positive spectrum in the 4D representation – which is related to the fact that their orbits in the 4D Dirac space \overline{M}_4 are spacelike rather than future timelike. We shall briefly return to this in Section 5.

4 Conformal Holography

4.1 Timelike Surfaces

The question arises to which extent one can recover a D -dimensional QFT from its restrictions. Clearly, in some form the higher-dimensional conformal symmetry group and its unitary representation must be present in the lower-dimensional theory. It is possible [16] to give a system of axioms on the inner symmetries of a lower-dimensional GCI CFT, which ensure that the theory can be extended to a higher-dimensional GCI CFT.

4.2 Lightfront Holography

The characteristic initial value problem for the Klein-Gordon operator in $D > 2$ dimensions consists in finding a solution to $(\square_x + m^2)\varphi_m(x) = 0$ with prescribed values $\varphi_N(x_+, x_\perp)$ of φ (as in Sect. 3.2) on the null (characteristic) hypersurface $N = \{x \in \mathbb{R}^4 : x^0 = x^1\}$ with sufficiently rapid decay.

A (unique?) solution is given in terms of the massive commutator function

$$C_m(x - y) = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \text{sign}(k^0) e^{-ikx}$$

by

$$\varphi_m(x) = -2i \int_N dy_+ d^2y_\perp C_m(x - y)|_{y_- = 0} \varphi_N(y_+, y_\perp). \quad (5)$$

Notice the fact that the kernel $C_m(x - y)|_{y_- = 0}$ solves the KG equation w.r.t. x , and restricts at $x_- = 0$ to

$$C_m(z)|_{z_- = 0} = \frac{i}{4} \text{sign}(z_+) \delta(z_\perp) \quad \Rightarrow \quad \partial_+ C_m(z)|_{z_- = 0} = \frac{i}{2} \delta(z_+) \delta(z_\perp).$$

(5) not only solves the classical initial value problem, but is indeed a relation between *quantum* fields in different dimensions: namely, if one takes for

$\varphi_N(y_+, y_\perp)$ the chiral free field with two-point function (4) and computes the two-point function of the r.h.s. of (5), one recovers the two-point function of the massive free field in \mathbb{R}^4 .

(5) is an adaptation of a similar formula used in [17] to pull back a state on the null future \mathcal{I}^+ of an asymptotically flat spacetime to a state on the bulk. The feature that a (free) field in Minkowski spacetime can be reconstructed from its restriction to the null hypersurface, which behaves like an infinite-component chiral conformal field, was first pointed out by Schroer [14, 18].

Interestingly enough, the massive free field of *any mass* can be recovered from the same conformal field theory on the lightfront, given by the free currents $j_n(x_+)$ ($n \in \mathbb{N}$), just by choosing the mass in the commutator function C_m . Schroer calls this “a different 4D spacetime organization of the same quantum substrate” (given by the chiral theory). Such a thing is possible because of the universality of the separable “inner” Hilbert space $L^2(\mathbb{R}^2)$.

4.3 2D Boundary Holography

In two dimensions, the presence of a boundary at $x^1 = 0$ leads to a reduction of the degrees of freedom because the boundary conditions imply that the left- and right-moving chiral fields are no longer independent but coincide with each other [19, 20]. In particular, the restriction of the chiral fields to the time axis (= the boundary) coincides with this chiral subtheory, while the restriction of non-chiral fields (not satisfying 2D Huygens locality) will in general be nonlocal on the time axis, but relatively local w.r.t. the chiral subtheory. The full CFT in the Minkowski halfspace $x^1 > 0$ can be recovered from the nonlocal boundary theory by a surprisingly simple algebraic construction [20].

Moreover, in a suitable state evaluated in the limit when all fields are localized “far away from the boundary”, the correlations converge to those of an associated full 2D CFT with two independent chiral subtheories [21]. The basic mechanism that restores the full 2D degrees of freedom (in particular, two chiral algebras) is the decoupling of left and right movers in the limit under consideration, due to the cluster property of the single chiral theory. The GNS reconstruction from this factorizing state then produces the tensor product of two chiral algebras.

5 4D Positivity

As mentioned before, the main open question concerning the double pole solutions of Section 2.2 is, whether they are compatible with Hilbert space positivity. To test positivity, one would like to split correlation functions that should be positive by Hilbert space positivity, into contributions that should be separately positive.

Such a decomposition is the partial wave expansion: a given correlation function splits into contributions

$$\langle D(x_4)C(x_3)\Pi_\lambda B(x_2)A(x_1)\rangle \quad (6)$$

where Π_λ are projections onto the irreducible representations $\lambda = (d, j_1, j_2)$ of the conformal group. Each term (6) is a coefficient times a partial wave = eigenfunction of differential operators corresponding to the three Casimir operators (quadratic, cubic, and quartic). Positivity requires in the simplest case, that all partial wave coefficients of correlations of the type $\langle ABBA \rangle$ in (6) must be non-negative, and associated Cauchy-Schwarz (CS) inequalities [2]. Even without knowing the six-point function, its mere existence imposes via CS inequalities further nontrivial constraints on the four- and two-point functions [22].

Let us notice here that positivity enters the analysis at several stages. First of all, the fields of the theory are subject to the unitarity bound [3]. Second, the condition that the operator product expansion of two fields does not involve fields below the unitarity bound, is reflected in bounds on the poles* in the variables ρ_{ij} [1], that were implicitly used throughout Section 2. While we regard these bounds as “kinematical”, the positivity of partial wave coefficients and the associated CS inequalities are “dynamical” constraints which are notoriously difficult to evaluate.

It should therefore be clear that we can only test *necessary* conditions for positivity throughout. Even so, the partial wave analysis is not practical for higher than four-point correlations, because the computation of the 4D partial waves seems out of reach. We therefore seek for simpler alternatives, that might give necessary conditions for positivity.

5.1 Positivity by Restriction

One option is to remark that restriction preserves Hilbert space positivity, since it only amounts to limits in the test function space, see also [11]. Hence, a 4D double pole structure must be rejected if its 2D restriction violates positivity.

Upon restriction, both the (tensor) fields will decompose into (subtensor) fields, and the irreps will split into irreps of the subgroup. Therefore, the restriction of a 4D partial wave is in general a sum of infinitely many 2D partial waves. To use this as a tool, it is necessary to understand the branching rules.

The branching of *representations* can be computed from the characters $\chi(s, x, y) = \text{Tr } s^{M_{05}}(xy)^{M_{12}}(x/y)^{M_{34}}$ of the representations, counting the multiplicities of the eigenvalues of the Cartan generators, see *e.g.*, [23]. For twist $\neq 2$

*Concerning these bounds, there were some inaccuracies in the admitted range of certain parameters around eq. (B.10) of [2]. That the partial waves are regular and the expansion formulae derived in [2] remain valid in the corrected parameter range, was checked in [26].

$$\chi_{d,j_1,j_2}^{4D}(s,x,y) = \frac{s^d \cdot \chi_{j_1}(x)\chi_{j_2}(y)}{(1-sx^{\frac{1}{2}}y^{\frac{1}{2}})(1-sx^{\frac{1}{2}}y^{-\frac{1}{2}})(1-sx^{-\frac{1}{2}}y^{\frac{1}{2}})(1-sx^{-\frac{1}{2}}y^{-\frac{1}{2}})},$$

where $\chi_j(x) = x^{-j} + x^{1-j} + \dots + x^{j-1} + x^j$. The restriction to 2D amounts to equating the parameters $x = y$. The branching is then given by the expansion into 2D characters

$$\chi_{h_+,h_-}^{2D}(p,q) = \chi_{h_+}(p) \cdot \chi_{h_-}(q) = \frac{p^{h_+}}{1-p} \cdot \frac{q^{h_-}}{1-q},$$

where $p = sx$ and $q = s/x$ couple to the chiral generators $L_0^\pm = \frac{1}{2}(M_{05} \pm M_{12})$. For instance, for the scalars $j_1 = j_2 = 0$, this gives the branching of representations

$$D_{d,0,0}^{4D}|_{2D} = \bigoplus_n (n+1) \cdot D_{(d+n)/2}^+ \otimes D_{(d+n)/2}^- \quad (7)$$

Since the representations are generated by corresponding fields from the vacuum, the multiplicity factor $n+1$ in this branching can be easily understood as counting the derivatives of the field in the restricted directions (leading to an increase of the dimension by one unit), in accord with the rules obtained from [6, 10]. In the general case, the factor $\chi_j(x)^2$ produces more terms corresponding to the decomposition of tensors into subtensors.

At twist $d - j_1 - j_2 = 2$ ($j_1, j_2 \neq 0$), there are subtractions in the characters reflecting the absence of some states due to the conservation laws (1). The factor $\chi_{j_1}(x)\chi_{j_2}(y)$ has to be replaced by $\chi_{j_1}(x)\chi_{j_2}(y) - s\chi_{j_1-\frac{1}{2}}(x)\chi_{j_2-\frac{1}{2}}(y)$, leading to a corresponding removal of some of the 2D subrepresentations.

The branching of *partial waves* follows a similar pattern. We consider here only scalar fields. Since a restricted scalar field is just another scalar field, only the decomposition of the projections in (6) matters. For the most symmetric four-point case when A, B, C, D are scalars of the same scaling dimension d , we found the following result.

Extracting a prefactor $(\rho_{12}\rho_{34})^{-d}$, the 4D partial waves depend only on the cross ratios $s = \frac{\rho_{12}\rho_{34}}{\rho_{13}\rho_{24}}$, $t = \frac{\rho_{14}\rho_{23}}{\rho_{13}\rho_{24}}$. For twist $2k$ and spin (tensor rank) $L = 2j_1 = 2j_2$ of the representation λ , they are given by [24]

$$\beta_{k,L}^{4D}(u,v) = \frac{uv}{u-v} (G_{k+L}(u)G_{k-1}(v) - (u \leftrightarrow v)),$$

where the ‘‘chiral’’ variables u, v are algebraic functions of s, t given by $s = uv$, $t = (1-u)(1-v)$, and $G_n(z) = z^n {}_2F_1(n, n; 2n; z)$. Upon restriction to $D = 2$, u and v become the chiral cross ratios $u = \frac{x_{12+}x_{34+}}{x_{13+}x_{24+}}$, $v = \frac{x_{12-}x_{34-}}{x_{13-}x_{24-}}$. The 2D partial waves of dimension $h_+ + h_-$ and helicity $h_+ - h_-$ are given by

$$\beta_{h_+,h_-}^{2D}(u,v) = G_{h_+}(u)G_{h_-}(v).$$

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Using repeatedly the identity $G_{n-1}(z) - \frac{1-z/2}{z}G_n(z) + c_n G_{n+1}(z) = 0$, where $c_n = \frac{n^2}{4(4n^2-1)}$, we found the recursion [25]

$$\beta_{k,L}^{4D} = \sum_{\substack{m,n \geq 0 \\ m+n=L}} \beta_{k+m,k+n}^{2D} + c_{k+L} \beta_{k+1,L}^{4D} + \sum_{\nu=1}^{[L/2]} (c_{k+L-\nu} - c_{k+\nu-1}) \beta_{k+\nu+1,L-2\nu}^{4D}.$$

For $L = 0$ or 1 , the last sum on the r.h.s. is empty. The 4D partial waves on the r.h.s. can be iteratively expanded by the same formula, giving all 2D partial waves of dimension $2k + L + 2r$ in the r -th step of the iteration:

$$\begin{aligned} \beta_{k,0}^{4D} &= \sum_{r \geq 0} c_k c_{k+1} \cdots c_{k+r-1} \beta_{k+r,k+r}^{2D}, \\ \beta_{k,1}^{4D} &= \sum_{r \geq 0} c_{k+1} c_{k+2} \cdots c_{k+r} \left(\beta_{k+r+1,k+r}^{2D} + \beta_{k+r,k+r+1}^{2D} \right). \end{aligned} \quad (8)$$

If $L \geq 2$, the last sum contains negative coefficients (because c_n is monotonously decreasing); but the iteration of the term $c_{k+L} \beta_{k+1,L}^{4D}$ contributes to the same 2D partial waves, making the total coefficients positive, *e.g.*,

$$\begin{aligned} \beta_{k,2}^{4D} &= \sum_{r \geq 0} c_{k+2} c_{k+3} \cdots c_{k+r+1} \\ &\times \left(\beta_{k+r+2,k+r}^{2D} + \frac{c_{k+r+1} + c_{k+r} - c_k}{c_{k+r+1}} \beta_{k+r+1,k+r+1}^{2D} + \beta_{k+r,k+r+2}^{2D} \right). \end{aligned}$$

Comparing (7) with (8), there seems to be a discrepancy, since the latter sum runs only over integer r , *i.e.*, half of the representations present in (8) are absent in the restricted partial wave. This teaches us that in order to “exhaust” the full content of representations in a restricted partial wave, one must also consider derivatives of the fields in the restricted directions, before restricting.

In order to extend this tool to six-point functions, one would need to know six-point partial waves. We do not know these partial waves, but it is clear that the Casimir eigenvalue equations are much easier to access in 2D than in 4D [25].

5.2 The Exotic Restriction (Continued)

Let us resume the discussion of Section 3.3. The generators L_0^\pm of the 2D conformal group embedded into the 4D conformal group are, in this case, M_{12} and M_{34} . Thus one should obtain the decomposition of representations by putting $s = 1$, and letting xy and x/y play the role of p and q before. It is then obvious that the expansion involves negative powers of p and q , reflecting the obvious

fact that M_{12} and M_{34} do not have positive spectrum in 4D positive energy representations. The expansion technique of the previous subsection fails in this situation.

More detailed analysis of the spectrum of the two chiral Casimir operators [25] indicates that the decomposition goes into a *continuum* of representations of the Möbius groups with positive and negative unbounded spectrum of L_0^\pm .

5.3 Characterization of Twist 2 Contributions

Another idea to isolate parts from correlation functions that must be separately positive, is to use the twist. This is a convenient “quantum number”, but not an eigenvalue of any polynomial function of the Casimir operators. Yet, as the discussion of biharmonic fields shows, the projection to the sum of all twist 2 representations

$$\langle \cdots \Pi_{\text{twist } 2} A'(x)A(y) \rangle = \sum_{\lambda: \text{twist}(\lambda)=2} \langle \cdots \Pi_\lambda A'(x)A(y) \rangle$$

is, after multiplication with $((x-y)^2)^{d-1}$, characterized by the very simple pair of differential equations (2). This suggests the following potential technique. We know that

$$\langle V(x, y) \Pi_{\text{twist } 2} C(x_3)B(x_2)A(x_1) \rangle = \langle V(x, y) C(x_3)B(x_2)A(x_1) \rangle \quad (9)$$

is a biharmonic function due to (2) for every biharmonic field V . Since by conformal invariance, correlation functions depend essentially only on the cross ratios, here regarded as “collective variables”, one may expect that the same information encoded in the wave operators \square_x and \square_y , can be encoded in a system of differential operators w.r.t. the variables x_1, x_2, x_3 , annihilating $\langle V C B A \rangle$. Then, under the reasonable hypothesis, that all biharmonic fields of the theory generate the entire twist 2 subspace of the Hilbert space, this would imply that the vector $\Pi_{\text{twist } 2} C B A \Omega$ solves the same equations, and so does the six-point correlation function

$$\langle A(x_6)B(x_5)C(x_4) \Pi_{\text{twist } 2} C(x_3)B(x_2)A(x_1) \rangle.$$

This information can be used to compute the form of this contribution, and to isolate the twist 2 part of a given six-point correlation $\langle A B C C B A \rangle$, because the higher twists are less singular. If the twist 2 part fails to be positive, the full six-point function is not positive. Ultimately, we would like to apply this strategy to six-point double pole structures which appear in correlations of the form $\langle V C C V \rangle$ [4].

As a first step towards this program, we have tested the idea on four-point functions [26]. So let C be the unit operator in (9). If A and $B = A'$ have the

same scaling dimension, it is obvious that the twist 2 projection selects the biharmonic field $V_{A'A}$, and it is also known that the wave operators w.r.t. x and y , if expressed in terms of the cross ratios s, t , are the same as the wave operators w.r.t. the arguments of $V_{A'A}(x_2, x_1)$. Hence, in this case the strategy works.

Less obvious is the case when $d_A \neq d_B$. The difference $d_B - d_A = 2n$ must be even by GCI, and we may assume $n > 0$. Writing

$$(x_{12}^2)^{d_A+n-1} \cdot \langle V(x, y)B(x_2)A(x_1) \rangle = f(x, y, x_2, x_1),$$

we found [26] that biharmonicity in x and y implies the pair of equations

$$[x_{12}^2 \partial_1 \cdot \partial_2 - 2(x_{12} \otimes x_{12}) \cdot (\partial_1 \otimes \partial_2) + 2(n-1)x_{12} \cdot \partial_2 + 2(n+1)x_{12} \cdot \partial_1] f = 0,$$

and

$$(\partial_1^{\otimes n})_{\text{traceless}} f = 0,$$

i.e., a pair of differential operators w.r.t. x_1 and x_2 characterizing “twist 2”, as desired. The first equation is actually equivalent to

$$\langle V(x, y)(\mathcal{C} - \lambda)B(x_2)A(x_1) \rangle = 0,$$

where \mathcal{C} is the quadratic Casimir operator and λ its eigenvalue in the scalar representation of dimension 2. Hence, the twist 2 contribution $\Pi_{\text{twist } 2} B(x_2)A(x_1)\Omega$ consists of a scalar part only. This is an independent proof of Lemma 5.2 in [2] which states that the only twist 2 contribution in the operator product expansion of two GCI scalar fields of different dimension is the scalar $d = 2$ representation. The second equation is equivalent to the statement that every correlation $(x_{12}^2)^{d_A+n-1} \langle \dots \Pi_{\text{twist } 2} B(x_2)A(x_1) \rangle$ is a homogenous *polynomial* in ρ_{1i} of order $n - 1$.

An illustrating free field example for $n = 2$ is the following. Let φ be the massless free field, and W_μ a conformal vector field of dimension $\Delta > 3$. Then $A = :W_\mu W^\mu:$ and $B = :[(\Delta - 3)W^\mu \partial_\mu \varphi - \varphi(\partial_\mu W^\mu)]^2:$ are conformal scalars of dimension $d_A = 2\Delta$ and $d_B = 2\Delta + 4$. The projection $\Pi_{\text{twist } 2}$ acting on $B(x_2)A(x_1)\Omega$ amounts to the contraction of all W fields. The result is $(x_{12}^2)^{-d_A-1}$ times the vector

$$(x_{12}^2 \square_2 - 4x_{12} \cdot \partial_2 + 8) : \varphi^2(x_2) : \Omega$$

which is indeed annihilated by the two differential operators above. Splitting any correlation $(x_{12}^2)^{d_A+1} \langle \dots B(x_2)A(x_1) \rangle$ into a part in the kernel of the two differential operators and a less singular part, uniquely selects this vector. (Incidentally, in this case, the first operator is sufficient to do the job.)

6 Conclusion

We have presented a number of ideas and new techniques which might be developed into useful tools for the analysis of globally conformal invariant correlation

functions, especially the problem of Hilbert space positivity of correlation functions that cannot arise from free fields. Various side aspects, concerning the relations between conformal QFT in four, two and one (chiral) dimensions were also discussed.

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Note. During the symposium, KHR became aware of the recent work by G. Mack [11], also presented on that occasion, which has some overlap with ours.

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