A Construction of Lie Superalgebras $B(m,n)$ and $D(m,n)$ from Triple Systems

N. Kamiya
Center for Mathematical Sciences, University of Aizu, 965-8580, Japan

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Abstract. It is well known that the concept of a triple system (= vector space equipped with a triple product $\langle xyz \rangle$) plays an important role in the construction of simple Lie algebras or superalgebras by means of the standard embedding associated with triple systems.

In this paper, we will give a standard embedding construction of Lie superalgebra of types of $P(n), Q(n)$ and $B(m, n), D(m, n)$ from triple systems as well as $G(3), F(4)$ and $D(2, 1, \alpha)$. Also, we will consider a Peirce decomposition of $(-1,-1)$-Freudenthal-Kantor triple systems.

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1 Introduction

One of the main object of study in this article is to provide examples of $(\varepsilon, -1)$-Freudenthal-Kantor triple systems and of Lie superalgebras associated with their triple systems.

It is known that the all simple Lie algebras $L$ have a decomposition of 5-graded Lie algebras as follows;

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

starting with a triple system, which has a triple product’s structure into the subspace component $L_1$ of $L$. And if $\dim L_{-2} = \dim L_2 = 1$, it is said to be a balanced triple system for $L_1$. Furthermore, by means of no using root systems and Cartan matrices, a property of 5-grading of Lie algebras is reduced from the property of triple systems equipped with 2nd order. This is one of the simple reasons for us to consider about the triple systems.

Generally speaking for our mathematical field (that is, nonassociative algebras), it seems that nonassociative algebras are rich in algebraic structures and mathematical physics. They provide an important common ground for various
branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry. That is, the concept of nonassociative algebras, which contain Jordan algebras (superalgebras) and Lie algebras (superalgebras), plays an important role in many mathematical and physical subjects [1–3, 6–8, 17, 25, 29, 31–34, 36–38]. We have determined that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems [12,20,28,35], in particular, by using the standard embedding method [23, 30, 36].

Describing our recent results in brief, we find the following:

- For the construction of simple Lie algebras, the generalized Jordan triple system of second order (that is, the \((-1, 1)\)-Freudenthal-Kantor triple system) is a useful concept [9–16, 21].
- For the construction of simple Lie superalgebras, the \((-1, -1)\)-Freudenthal-Kantor triple system is a useful concept [5, 23, 24, 26].
- For the construction of Jordan superalgebras, the \(\delta\)-Jordan-Lie triple system is a useful concept [27, 28, 35].

Our purpose is to propose a unified structural theory for triple systems.

We shall be concerned with algebras and triple systems which are finite dimensional over a field \(\Phi\) of characteristic 0 unless otherwise specified.

2 Freudenthal-Kantor Triple Systems

We will define and provide some relevant facts about a \((\varepsilon, \delta)\)-Freudenthal-Kantor triple system [9–16, 23] in order to make this paper as comprehensive as possible.

For \(\varepsilon = \pm 1\) and \(\delta = \pm 1\), a vector space \(U(\varepsilon, \delta)\) over a field \(\Phi\) of \(\text{ch } \Phi \neq 2, 3\) with a triple product \(L(x, y)z := \langle x y z \rangle\) is called a \((\varepsilon, \delta)\)-Freudenthal-Kantor triple system if

\[
\langle abc \rangle \in U(\varepsilon, \delta) \quad (1.0)
\]

\[
\langle ab\langle cde \rangle \rangle = \langle \langle abc \rangle de \rangle + \varepsilon \langle c\langle bad \rangle e \rangle + \langle cd\langle abc \rangle \rangle \quad (1.1)
\]

\[
K(\langle abc \rangle, d) + K(c, \langle abd \rangle) + \delta K(a, K(c, d)b) = 0, \quad (1.2)
\]

where \(K(a, b)c = \langle abc \rangle - \delta\langle bca \rangle\).

In particular, it is said to be a generalized Freudenthal-Kantor triple system if the conditions (1.0) and (1.1) are satisfied.

Furthermore, it is said to be balanced if the \((\varepsilon, \delta)\)Freudenthal-Kantor triple system has a property such that

\[
K(a, b) = \langle a\vert b \rangle Id, \quad \langle a\vert b \rangle = -\varepsilon \langle b\vert a \rangle \quad \in \Phi.
\]
Remark. We note that the notation of the \((-1,1\))-Freudenthal-Kantor triple system is equal to that of the generalized Jordan triple system of second order due to I.L. Kantor, thus this triple system is closely related to the structurable algebra due to B.N. Allison [1, 10]. Also this algebra was generalized by the author (to see [14, 17, 19]. In a special case, we shall restrict the case of \(K(a, b) = 0\) for all \(a, b \in U(\varepsilon, \delta)\). We call this a \((\varepsilon, \delta)\)-Jordan triple system. That is, a \((\varepsilon, \delta)\)-Jordan triple system is defined as
\[
\langle abc \rangle = \delta \langle cba \rangle \tag{1.3}
\]
\[
[L(a, b), L(c, d)] = L(\langle abc \rangle, d) + \varepsilon L(c, \langle bad \rangle), \tag{1.4}
\]
where \([A, B] = AB - BA\).
For \(\varepsilon = -1, \delta = 1\) (resp. \(\varepsilon = 1, \delta = -1\)) of a \((\varepsilon, \delta)\)-Jordan triple system, it is said to be a Jordan triple system (resp. anti-Jordan triple system).

Remark. For \(\varepsilon = -1, \delta = 1\), it is well known that the above triple systems defined by the relations of (1.3) and (1.4) are closely related to symmetric bounded domains or symmetric \(R\)-spaces based on certain conditions of a geometrical phenomenon [23, 31].

Example 1.1. Let \(W\) be a set of the matrix \(\text{Mat}(m, n : \Phi)\). Then \(\langle xyz \rangle = x^t yz + \delta z^t yx\) defines a \((-1, \delta)\)-Jordan triple system on \(W\), where \(\delta = \pm 1\), and \(^t x\) denotes the transpose matrix of \(x\).

We recall a \(\delta\)-Lie triple system that satisfies the following identities \([10, 23, 30, 35]\):
\[
[xyz] = -\delta[yxz] \tag{L1}
\]
\[
[xyz] + [yzx] + [zxy] = 0 \tag{L2}
\]
\[
[L(x, y), L(z, w)] = L([xyz], w) + L(z, [xyw]) \tag{L3}
\]
where \([x, y]z = [xyz]\).
We often use the notation \([xyz]\) with respect to the product of Lie triple systems instead of \(\langle xyz \rangle\), or \{xyz\} as the traditional notation.

For \(\delta = 1\) (resp. \(\delta = -1\)), we call it a Lie triple system (resp. anti-Lie triple system).

Proposition 1.1. Let \((U, [xyz])\) be a \(\delta\)-Lie triple system. Then \((U, [xyz])\) has the structure of a generalized Freudenthal-Kantor triple system with \(\varepsilon = -\delta\).

Also we recall a \(\delta\)-Jordan-Lie triple system as a variation of the \(\delta\)-Lie triple system [27, 35]:
\[
\langle xyz \rangle = -\delta \langle yxz \rangle \tag{J1}
\]
\[
\langle xyz \rangle + \langle yzx \rangle + \langle zxy \rangle = 0 \tag{J2}
\]
\[
\langle uv\langle xyz \rangle \rangle = \langle \langle uvx \rangle yz \rangle + \langle x\langle uvy \rangle z \rangle + \delta \langle xy\langle uvz \rangle \rangle \tag{J3}
\]
**Remark.** [23, 35] For a \( \delta \)-Lie triple system \( T \) and \( \delta \)-Jordan-Lie triple system \( T' \), we put

\[
L(T) := L(T, T) \oplus T, \quad J(T') := L(T', T') \oplus T',
\]

where \( L(T, T) := \{ L(x, y) \mid x, y \in T \} \), \( L(T', T') := \{ L(x, y) \mid x, y \in T' \} \).

Then we note that our triple systems \( T \) and \( T' \) may construct the standard embedding algebras as follows:

- \( \delta \)-Lie triple system \( T \rightarrow \) Lie algebra or superalgebra \( L(T) \)
- \( \delta \)-Jordan-Lie triple system \( T' \rightarrow \) Jordan algebra or superalgebra \( J(T') \).

For the \( \delta \)-Lie triple systems associated with \( (\varepsilon, \delta) \)-Freudenthal-Kantor triple systems, we have the following.

**Proposition 1.2.** [11, 23] Let \( U(\varepsilon, \delta) \) be a \((\varepsilon, \delta)\)-Freudenthal-Kantor triple system. If \( P \) is a linear transformation of \( U(\varepsilon, \delta) \) such that \( P \langle xyz \rangle = \langle PxPyPz \rangle \) and \( P^2 = -\varepsilon \delta Id \), then \( (U(\varepsilon, \delta), [-, -, -]) \) is a Lie triple system for the case of \( \delta = 1 \) and an anti-Lie triple system for the case of \( \delta = -1 \) with respect to the product

\[
[xyz] := \langle xPyz \rangle - \delta \langle yPxz \rangle + \delta \langle xPzy \rangle - \langle yPzx \rangle.
\]

**Corollary.** Let \( U(\varepsilon, \delta) \) be a \((\varepsilon, \delta)\)-Freudenthal-Kantor triple system. Then the vector space \( T(\varepsilon, \delta) := U(\varepsilon, \delta) \oplus U(\varepsilon, \delta) \) becomes a Lie triple system for the case of \( \delta = 1 \) and anti-Lie triple system for the case of \( \delta = -1 \) with respect to the triple product defined by

\[
\begin{bmatrix}
(a) & (c) & (e) \\
(b) & (d) & (f)
\end{bmatrix} =
\begin{bmatrix}
L(a, d) - \delta L(c, b) & \delta K(a, c) \\
-\varepsilon K(b, d) & \varepsilon (L(d, a) - \delta L(b, c))
\end{bmatrix}
\begin{bmatrix}
e \\
f
\end{bmatrix}.
\]

**Proposition 1.3.** Let \( V \) be an anti-Jordan triple system. Then, \( V \oplus V \) becomes an anti-Lie triple system with respect to the product defined by

\[
\begin{bmatrix}
(a) & (c) & (e) \\
(b) & (d) & (f)
\end{bmatrix} =
\begin{bmatrix}
L(a, d) + L(c, b) & 0 \\
0 & L(d, a) + L(b, c)
\end{bmatrix}
\begin{bmatrix}
e \\
f
\end{bmatrix}.
\]

From these results, it follows that the vector space

\[
L(V) := \text{Inn Der} \ T \oplus T \ (= L(T, T) \oplus T),
\]

where \( T \) is a \( \delta \) Lie triple system, and \( \text{Inn Der} \ T := \{ L(X, Y) \mid X, Y \in T \} \text{span} \), make a Lie algebra \((\delta = 1)\) or Lie superalgebra \((\delta = -1)\) by

\[
[D + X, D' + X'] = [D, D'] + L(X, X') + DX' - D'X.
\]
We denote by $L(\epsilon, \delta)$ the Lie algebras or Lie superalgebras obtained from these construction associated with $U(\epsilon, \delta)$ and call these algebras a standard embedding. A $(\epsilon, \delta)$-Freudenthal-Kantor triple system $U(\epsilon, \delta)$ is said to be unitary if the linear span $k$ of the set $\{K(a, b) | a, b \in U(\epsilon, \delta)\}$ contains the identity endomorphism Id.

Proposition 1.4. [11, 13] For a unitary $(\epsilon, \delta)$ Freudenthal-Kantor triple system $U(\epsilon, \delta)$ over $\Phi$, let $T(\epsilon, \delta)$ be the Lie or anti Lie triple system and $L(\epsilon, \delta)$ be the standard embedding Lie algebra or superalgebra associated with $U(\epsilon, \delta)$. The following are equivalent:

a) $U(\epsilon, \delta)$ is simple,

b) $T(\epsilon, \delta)$ is simple,

c) $L(\epsilon, \delta)$ is simple.

For these standard embedding Lie algebras or superalgebras $L(\epsilon, \delta)$, we have the 5 grading subspaces as follows:

$L(\epsilon, \delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$

where $U(\epsilon, \delta) = L_{-1}, T(\epsilon, \delta) = L_{-1} \oplus L_1, k = \{K(a, b)\}_{\text{span}} = L_{-2}$.

3 Construction of $P(n)$ and $Q(n)$

We will discuss examples of the standard embedding Lie superalgebra associated with anti-Lie triple systems and anti-Jordan triple systems by means of the argument given in Section 1.

Lemma 2.1. Let $A$ be a symmetric matrix and $B$ be an anti-symmetric matrix. Then $\text{tr}(AB) = 0$, where $\text{tr}$ is the trace form.

Proof. By the property of trace, we obtain

$$\text{tr}(AB) = \text{tr}^t(AB) = \text{tr}(t^tB^tA) = -\text{tr}(BA).$$

On the other hand, we have

$$\text{tr}(AB) = \text{tr}(BA).$$

Thus we get

$$\text{tr}(AB) = -\text{tr}(AB),$$

and so $\text{tr}(AB) = 0$.

Theorem 2.2. Let $V$ be a set of the matrix $\begin{pmatrix} 0 & x \\ x' & 0 \end{pmatrix}$, where $x$ is a symmetric matrix of $(n+1) \times (n+1)$ and $x'$ is an anti-symmetric matrix of $(n+1) \times (n+1)$. Then $V$ is an anti-Jordan triple system with respect to the product defined by

$$\{XYZ\} := \begin{pmatrix} 0 & xy'z - zy'x \\ x'yz' - z'y'x' & 0 \end{pmatrix}.$$
Proof. By straightforward calculations, we can obtain the proof. Thus we omit it.

From Proposition 1.2, (a special case, $\varepsilon = 1, \delta = -1, K(a, b) \equiv 0$), we have the following.

**Theorem 2.3.** Let $(V, \{XYZ\})$ be an anti-Jordan triple system. Then $V$ is an anti-Lie triple system with respect to the product defined by

$$[XYZ] = \{XYZ\} + \{YXZ\}.$$ 

Let $V$ be a set of the matrix described above in Theorem 2.2.

We can state that the standard embedding Lie superalgebra associated with $V$ is as follows.

$$P(n) := \left\{ \begin{pmatrix} a & b \\ c & -^ta \end{pmatrix} \mid \text{tr} \ a = 0, \ b: \text{sym.} \ c: \text{anti-sym.} \right\} = L(V, V) \oplus V.$$ 

Thus this $P(n)$ may be constructed from the above concept and the results in Section 1.

In the final part of this section, we will discuss the construction of $Q(n)$.

**Theorem 2.4.** Let $V$ be a set of \( \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \), where $b$ is a \((n+1) \times (n+1)\) matrix.

Then $V$ is an anti-Jordan triple system with respect to the triple product $\{XYZ\}$ defined by

$$\{XYZ\} := XYZ - ZYX,$$

for all $X, Y, Z \in V$.

Proof. It is clear. We omit the proof.

Combining above Theorem 2.4 with Theorem 2.3, we obtain the following result:

$$L(V) = L(V, V) \oplus V = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\}_{\text{span}} \oplus \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \right\}_{\text{span}}.$$ 

By this construction, we can make the Lie superalgebra of type $Q(n)$.

That is, we note that $L(V)/C$ is the simple Lie superalgebra of type $Q(n)$, where $C$ is the center of $\left\{ L(x, y) \right\}_{\text{span}}$ as the Lie superalgebra of $L(V)$.

In fact, since $\left\{ \begin{pmatrix} 1d & 0 \\ 0 & 1d \end{pmatrix} \right\} \in L(V, V)$, $\left\{ \begin{pmatrix} 1d & 0 \\ 0 & 1d \end{pmatrix} \right\}_{\text{span}}$ is an ideal of $L(V)$ and $C = \left\{ \begin{pmatrix} 1d & 0 \\ 0 & 1d \end{pmatrix} \right\}_{\text{span}}$. 

133
Furthermore, for the structure of the anti-Lie triple system $V$, if \[
\begin{pmatrix}
0 & b \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & c \\
0 & 0
\end{pmatrix}
\]
and \[
\begin{pmatrix}
0 & d \\
d & 0
\end{pmatrix} \in V \text{ (trace } b = tr c = tr d = 0 \text{)}
\]
then
\[
\begin{pmatrix}
0 & b \\
0 & 0
\end{pmatrix}\begin{pmatrix}
0 & c \\
0 & 0
\end{pmatrix}\begin{pmatrix}
0 & d \\
0 & 0
\end{pmatrix} = 
\begin{pmatrix}
0 & \dfrac{bc + cb}{d} - d(bc + cb) \\
(bc + cb)d - d(bc + cb) & 0
\end{pmatrix} \in V,
\]
where \( tr[(bc + cb)d - d(bc + cb)] = 0 \).

**Remark.** We note that the results in this section has a overlap to cowork with S. Okubo in [26].

### 4 Examples of (-1,-1) Freudenthal-Kantor triple systems

In this section, we will consider the standard embedding Lie superalgebras of the $B(m, n)$ and $D(m, n)$ types associated with an anti-Lie triple system and a (-1,-1) Freudenthal-Kantor triple system.

**Theorem 3.1.** Let $U$ be a vector space of $\text{Mat}(k, n; \Phi)$. Then the space $U$ is a unitary (-1,-1) Freudenthal-Kantor triple system with respect to the triple product
\[
\langle xyz \rangle = z^t yx + y^t xz - x^t yz.
\]
where $^t x$ denotes the transpose matrix.

For this triple system, by straightforward calculations, from the results in section 1, we have the following:

(i) $k = 2m, (m \geq 2)$

$L(U) \cong D(m, n)$ type's Lie superalgebra

and

$$\dim L(U) = 2(n + m)^2 - m + n,$$

(ii) $k = 2m + 1, (m \geq 0)$

$L(U) \cong B(m, n)$ type's Lie superalgebra

and

$$\dim L(U) = 2(n + m)^2 + 3n + m,$$

That is, sumarizing these, we have the following.

**Theorem 3.2.** Let $U$ be the triple system of same as described in Theorem 3.1 and $L(U)$ be the standard embedding Lie superalgebras associated with $U =$.
A Construction of Lie Superalgebras ...

$\text{Mat}(k, n; \Phi)$. Then $L(U)$ are Lie superalgebras of type $D(m, n)$ or $B(m, n)$ if $k=2m$ or $k=2m+1$, respectively.

For a bilinear trace form of a $(-1,-1)$-Freudenthal-Kantor triple system, we have a formula as follows [13];

$$\gamma(x, y) := \frac{1}{2} \text{trace}\{2(R(x, y) + R(y, x)) + L(x, y) + L(y, x)\},$$

where $R(x, y)z = \langle xzy \rangle$, and $L(x, y)z = \langle xyz \rangle$.

Thus in this case $U = \text{Mat}(k, n; \Phi)$ of the above Theorem 3.1, by straightforward calculations, we obtain the identity;

$$\gamma(x, y) = c_{x,y}(2n + 2 - k),$$

where $c_{x,y}$ is a constant element in $\Phi$ with dependent $x, y \in U$.

This implies that the trace form $\gamma(x, y)$ is degenerate if $m = n + 1$, (the case of $k = 2m$). Thus this fact has a relation to the degenerate property of the Killing form of the Lie superalgebra $D(n + 1, n)$.

Remark. For the construction of balanced types of Lie algebras and superalgebras, that is, in the case of $\dim k = \dim L_1 = \dim L_2 = 1$, we are proposing to refer in [5, 12, 14, 24].

For the detail contents in this section, we will discuss it in a future paper.

5 Peirce decompositions of triple systems

In this section, we shall announce a Peirce decomposition of $(-1,-1)$ Freudenthal-Kantor triple systems. In our previous work [20], we have studied the Peirce decomposition of the generalized Jordan triple system $U$ of second order by employing a tripotent element $e$ of $U$, (the tripotent element means $\{eee\} = e$).

The Peirce decomposition of $U$ is described as follows:

$$U = U_{00} \oplus U_{11} \oplus U_{12} \oplus U_{21} \oplus U_{-10} \oplus U_{01} \oplus U_{22} \oplus U_{13},$$

where $L(a) = \{eaa\} = \lambda a$, and $R(a) = \{ae\} = \mu a$ if $a \in U_{\lambda \mu}$.

In particular, if the tripotent element is a left unit (the left unit element $e$ means $eex = x, \forall x \in U$), then we have

$$U = U^+_{11} \oplus U^-_{11} \oplus U^+_{13} \oplus U^-_{13},$$

where $Q(x) = \pm x$ if $x \in U^\pm_{11}$, and $Q(x) = \pm 3x$ if $x \in U^\pm_{13}$.

On the other hand, for the Peirce decomposition of a Jordan triple system $U$, it is well known that

$$U = U_{00} \oplus U_{11} \oplus U_{11}, \text{ (only 3-component's decomposition)}.$$
Remark. For the balanced GJTSs of 2nd order of the exceptional types \( G_2, F_4, E_6, E_7 \) and \( E_8 \) associated with exceptional simple Lie algebras, we will consider their Peirce decompositions in the forthcoming paper [16].

Remark. For the balanced GJTSs of 2nd order, one study has been considered from a geometrical approach (see [4]), that is, he conducted the correspondence of quaternionic structures on symmetric spaces with balanced Freudenthal-Kantor triple systems. Thus it seems that our decompositions are useful for the detail’s characterization in supermanifold or geometry.

Remark. It seems that this field of nonassociative algebras is a very important subject in mathematical physics and differential geometry as well as the characterization and construction of Lie algebras, Lie superalgebras and Yang-Baxter equations (for example, [22, 34–36]).

As next part of our study of triple systems and Lie superalgebras in section 3, in forthcoming paper [18], we will discuss with the Peirce decomposition of triple systems associated with their superalgebras, however we will only give a brief statement as follows.

**Theorem 4.1.** Let \( U \) be a \((-1, -1)\) Freudenthal-Kantor triple system. Then we have a decomposition

\[
U = U_{00} \oplus U_{01} \oplus U_{1,-1} \oplus U_{1,1},
\]

where \( U_{i,j} \) denote the set of \( \{ x \in U | L(e,e)x = ix, R(e,e)x = jx \} \).

An example of the decomposition of \( B(m,n) \)’s and \( D(m,n) \)’s cases described in section 3 is as follows.

**Example 4.1.** For the triple system given in Theorem 3.1, we have a decomposition as follows. Let \( U = \text{Mat}(k,n;\Phi) \) be the triple system defined by

\[
\langle xyz \rangle = z^t y x + y^t x z - x^t y z.
\]

Here, let \( k \geq n > l \) and we set

\[
e := \begin{pmatrix} E_l & 0 \\ 0 & 0 \end{pmatrix} \cdots (k,n)\text{matrix}, E_l \text{ is } (l,l) \text{ identity matrix,}
\]

\[
x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ A : (l,l), B : (l,n-l), C : (k-l,l), D : (k-l,n-l) \text{ matrix.}
\]

Then we have

\[
x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A - A^t / 2 & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} A + A^t / 2 & 0 \\ 0 & 0 \end{pmatrix}.
\]

\[
\in U_{00} \oplus U_{01} \oplus U_{1,-1} \oplus U_{11}.
\]
A Construction of Lie Superalgebras ...

In particular, for this triple system, let \( k \geq n = l \), and we set
\[
e := \begin{pmatrix} E_l \\ 0 \end{pmatrix} \cdots (k, l) \text{ matrix.}
\]

Then we have
\[
eex = x \text{ for all } x \in U, \text{ and } xee = e^t ex + e^t xe - x,
\]

thus by straightforward calculations, we may obtain a decomposition,
\[
x = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A + \frac{1}{2} A^t \\ B \end{pmatrix} = \begin{pmatrix} A - \frac{1}{2} A^t \\ B \end{pmatrix} \in U_{11} \oplus U_{1,-1}.
\]

In final comments of this section, we give several simple examples of Peice decompositions of triple systems as follows.

In particular, for a balanced (-1,-1) Freudenthal-Kantor triple system, that is, in a triple system equipped with \( xxy = xyx = (\langle x|x \rangle y, \text{ and } \langle x|y \rangle = (y|x) \), we have the following.

**Proposition 4.2.** Let \( U \) be a balanced (-1,-1) Freudenthal-Kantor triple system. Then the decomposition is given by
\[
U = U_{11} \oplus U_{1,-1},
\]

where \( U_{11} = \{ x \in U| R(x) = x \} \) and \( U_{1,-1} = \{ x \in U| R(x) = -x \} \).

On the other hand, for a quadratic triple system [29], that is, in a triple system equipped with \( xxy = yxx = (\langle x|x \rangle y \), and \( \langle x|y \rangle = (y|x) \), we have the following.

**Proposition 4.3.** Let \( U \) be a quadratic triple system. Then the decomposition is given by
\[
U = U^+_{11} \oplus U^-_{11},
\]

where, \( U^+_{11} = \{ x \in U| eex = x \} \) and \( U^-_{11} = \{ x \in U| eex = -x \} \).

**Remark.** For a G.J.T.S. of 2nd order (= (-1,1) F-K.t.s.) equipped with \( xyz = (\langle y|z \rangle x \) and \( \langle x|y \rangle = (y|x) \), we have
\[
x = x - \langle x|e \rangle e + \langle x|e \rangle e
\]

that is,
\[
U = U_{01} \oplus U_{11},
\]

where, \( U_{01} = \{ x \in U| eex = 0 \} \) and \( U_{11} = \{ x \in U| ee = x \}, \langle e|e \rangle = 1 \).

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N. Kamiya

References


138
A Construction of Lie Superalgebras ...