Analytic Formulae for the Matrix Elements of the Transition Operators in the Symplectic Extension of the Interacting Vector Boson Model

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Abstract. The tensor properties of all the generators of Sp(12,R) - the group of dynamical symmetry of the Interacting Vector Boson Model (IVBM), are given with respect to the reduction chain Sp(12,R) ⊃ U(6) ⊃ U(3) × U(2) ⊃ O(3) × (U(1) × U(1)). Matrix elements of the basic building blocks of the model are evaluated in symmetry adapted basis along the considered chain. As a result of this, the analytic form of the matrix elements of any operator in the enveloping algebra of the Sp(12,R), defining a certain transition operator, can be calculated. The procedure allows further applications of the symplectic IVBM for the description of transition probabilities between nuclear collective states.

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1 Introduction

In the algebraic models the use of the dynamical symmetries defined by a certain reduction chain of the group of dynamical symmetry, yields exact solutions for the eigenvalues and eigenfunctions of the model Hamiltonian, which is constructed from the invariant operators of the subgroups in the chain.

Something more, it is very simple and straightforward to calculate matrix elements of transition operators between the eigenstates of the Hamiltonian, as both – the basis states and the operators, can be defined as tensor operators in respect to the considered dynamical symmetry. Then the calculation of matrix elements is simplified by the use of the respective generalization of the Wigner-Eckart theorem. By definition such matrix elements give the transition probabilities between the collective states attributed to the basis states of the Hamiltonian. The comparison of the experimental data with the calculated transition probabilities
Analytic Formulae for the Matrix Elements...

is one of the best tests of the validity of the considered algebraic model. With the aim of such applications of the rotational limit of symplectic extension of IVBM, we develop in this paper a practical mathematical approach for explicit evaluation of the matrix elements of transitional operators in the model.

The algebraic IVBM was developed [1] initially for the description of the low lying bands of the well deformed even-even nuclei [2]. Recently this approach was adapted to incorporate the newly observed higher collective states, both in the first positive and negative parity bands [3] by considering the basis states as "yrast" states for the different values of the number of bosons \( N \), that built them. This was achieved by extending the dynamical symmetry group \( U(6) \) to the noncompact \( Sp(12,R) \). The excellent results obtained for the energy spectrum require a further investigation of the transition probabilities in the framework of the generalized IVBM with \( Sp(12,R) \) as a group of dynamical symmetry. Thus in the present work we consider the tensor properties of the algebra generators (Section 2.) in respect to the reduction chain

\[
Sp(12,R) \supset U(6) \supset U(3) \times U(2) \supset O(3) \times (U(1) \times U(1)),
\]

and also classify the basis states (Section 3.) by the quantum numbers corresponding to the irreducible representations of its subgroups. In this way we are able to define the transition operators between the basis states and then to evaluate analytically their matrix elements (Section 4.).

2 Tensor Properties of the Generators of the \( Sp(12,R) \) Group

The basic building blocks of the IVBM [1] are the creation and annihilation operators of the vector bosons \( u^+_m(\alpha) \) and \( u_m(\alpha) \) \( (m = 0, \pm 1; \alpha = \pm \frac{1}{2}) \), which can be considered as components of a 6-dimensional vector, which transform according to the fundamental \( U(6) \) irreducible representations \([1,0,0,0,0,0]_6 \equiv [1]_6 \) and its conjugated \([0,0,0,0,-1]_6 \equiv [1]_6^* \), respectively. These irreducible representations become reducible along the chain of subgroups (1) defining the dynamical symmetry of the rotational limit of the model [2]. This means that along with the quantum number characterizing the representations of \( U(6) \), the operators are also characterized by the quantum numbers of the subgroups of chain (1).

The only possible representation of the direct product of \( U(3) \times U(2) \) belonging to the representation \([1]_6 \) of \( U(6) \) is \([1]_3[1]_2 \), i.e., \([1]_6 \equiv [1]_3[1]_2 \). According to the reduction rules for the decomposition \( U(3) \supset O(3) \), the representation \([1]_3 \) of \( U(3) \) contains the representation \([1]_3 \) of the group \( O(3) \) giving the angular momentum of the bosons \( l = 1 \) with a projection \( m = 0, \pm 1 \). The representation \([1]_2 \) of \( U(2) \) defines the "pseudospin" of the bosons \( T = \frac{1}{2} \), whose projection is given by the corresponding representation of \( (U(1) \times U(1)) \), i.e., \( \alpha = \pm \frac{1}{2} \). In this way the creation and annihilation operators \( u^+_m(\alpha) \) and \( u_m(\alpha) \) are defined.
as irreducible tensors along the chain (1), and the used phase convention and commutation relations are the following [4]:

\[
\begin{align*}
\left( u_{12}^{[1]6} \right)^{\alpha} = u_{[1]1}[1]2^{\alpha \mu} \left( -1 \right)^{m+12-\alpha} u_{[1]1}[1]2^{\alpha \mu} - m - \alpha \\
\left[ u_{[1]1}[1]2^{\alpha \mu}, u_{[1]1}[1]2^{\alpha \mu} \right] = \delta_{m,n} \delta_{\alpha,\beta}
\end{align*}
\]

Initially the generators of the symplectic group Sp(12, R) were written as double tensors [5] with respect to \(O(3) \supset O(2)\) and \(U(2) \supset U(1)\) reductions

\[
A_{LM}^{T T_0} = \sum_{m,n} \sum_{\alpha, \beta} C_{LM}^{1[1]6} C_{TT_0}^{12 \alpha \beta} u_{[1]1}[1]2^{\alpha \mu} u_{[1]1}[1]2^{\alpha \mu},
\]

\[
F_{LM}^{T T_0} = \sum_{m,n} \sum_{\alpha, \beta} C_{LM}^{1[1]6} C_{TT_0}^{12 \alpha \beta} u_{[1]1}[1]2^{\alpha \mu} u_{[1]1}[1]2^{\alpha \mu},
\]

\[
G_{LM}^{T T_0} = \sum_{m,n} \sum_{\alpha, \beta} C_{LM}^{1[1]6} C_{TT_0}^{12 \alpha \beta} u_{[1]1}[1]2^{\alpha \mu} u_{[1]1}[1]2^{\alpha \mu}.
\]

Further they can be defined as irreducible tensor operators according to the whole chain (1) of subgroups and expressed in terms of (3), (4), and (5)

\[
\begin{align*}
A_{[1]6}^{[1]6} \quad & \quad A_{[1]6}^{[1]6} = C_{[1]6}[1]6 C_{[1]6}[1]6 A_{TT_0}^{LM}, \\
F_{[1]6}^{[1]6} \quad & \quad F_{[1]6}^{[1]6} = C_{[1]6}[1]6 C_{[1]6}[1]6 F_{TT_0}^{LM}, \\
\end{align*}
\]

where, according to the lemma of Racah [6], the Clebsch-Gordan coefficients along the chain are factorized by means of isoscalar factors (IF), defined for each step of decomposition (1). It should be pointed out [4] that the \(U(6) - C_{[1]6}[1]6 C_{[1]6}[1]6\) and \(U(3) - C_{[1]6}[1]6 C_{[1]6}[1]6\) IF's, entering in (6), (7) and (8), are equal to \(\pm 1\) and their values, are taken into account in what follows.

The tensors (6), transform according to the direct product \([\chi]_6\) of the corresponding \(U(6)\) representations \([1]_6\) and \([1]_6^*\) [4], namely

\[
[1]_6 \times [1]_6^* = [1, -1]_6 + [0]_6
\]

where \([1, -1]_6 = [2, 1, 1, 1, 1, 0]_6\) and \([0]_6 = [1, 1, 1, 1, 1, 1]_6\) is the scalar \(U(6)\) representation. Further we multiply the two conjugated fundamental representations of \(U(3) \times U(2)\)

\[
\begin{align*}
[1]_3[1]2 \times [1]_3[1]2^* & = ([1]_3 \times [1]_3^*)[1]_2 \times [1]_2^* \\
& = ([210]_3 \oplus [1, 1, 1]) \times ([2, 0]_2 \oplus [1, 1]) \\
& = [210]_3[2]_2 \oplus [210]_3[0]_2 \oplus [0]_3[2]_2 \oplus [0]_3[0]_2.
\end{align*}
\]
The tensors resulting decomposition (10) belong to the $[1, -1]_6$ of $U(6)$ and the last one to $[0]_6$. Introducing the notations $u_i^+ (\frac{1}{2}) = p_i^+$ and $u_i^+ (-\frac{1}{2}) = n_i^+$, the scalar operator

$$A_{[0]_6}^{[0]_6} = \frac{1}{\sqrt{2}} \sum_m C_{1m}^{00} \left(p_m^+ p_{-m}^* + n_m^+ n_{-m}^*\right)$$

has the physical meaning of the total number of bosons operator $\hat{N} = \hat{N}_p + \hat{N}_n$, where $\hat{N}_p = \sum p_m^+ p_m$, $\hat{N}_n = \sum n_m^+ n_m$, and is obviously the first order invariant of all the unitary groups $U(6), U(3)$ and $U(2)$. Hence it reduces them to their respective unimodular subgroups $SU(6), SU(3)$ and $SU(2)$. Something more, the invariant operator $(-1)^\hat{N}$, decomposes the action space $H$ of the $Sp(12, R)$ generators to the even $H_+$ with $N = 0, 2, 4, \ldots$, and odd $H_-$ with $N = 1, 3, 5, \ldots$, subspaces of the boson representations of $Sp(12, R)$ [7].

The $U(3)$ irreps $[\lambda]_3$ are shorthand notations of $[n_1, n_2, n_3]_3$, and are expressed in terms of Elliott’s notations [8] $(\lambda, \mu)$ with $\lambda = n_1 - n_2, \mu = n_2 - n_3$, so in (10) we have $[210]_3 = (1, 1)$ and $[0]_3 = (0, 0)$. The corresponding values of $L$ from the $SU(3) \supset O(3)$ reduction rules are $L = 1, 2$ in the $(1, 1)$ irrep and $L = 0$ in the $(0, 0)$. The values of $T$ are 1 and 0 for the $U(2)$ irreps $[2]_2$ and $[0]_2$ respectively. Hence, the $U(2)$ irreps in the direct product distinguish the equivalent $U(3)$ irreps that appear in this reduction, and there is not degeneracy. The tensors with $T = 0$ correspond to the $SU(3)$ generators

$$A_{[210]_3[0]_2}^{[1-1]_6} = \frac{1}{\sqrt{2}} \sum_{m, k} C^{1M}_{1m1k} \left(p_m^+ p_k + n_m^+ n_k\right)$$

$$A_{[210]_3[0]_2}^{[1-1]_6} = \frac{1}{\sqrt{2}} \sum_{m, k} C^{2M}_{1m1k} \left(p_m^+ p_k + n_m^+ n_k\right)$$

representing the components of the angular $L_3\ell$ (12) and Elliott’s quadrupole $Q_3\ell$ momentum (13) operators.

The tensors

$$A_{[0]_6[2]_2}^{[1-1]_6} = \sqrt{\frac{3}{2}} \sum_m p_m^+ n_m \sim T_1,$$

$$A_{[0]_6[2]_2}^{[1-1]_6} = -\sqrt{\frac{3}{2}} \sum_m n_m^+ p_m \sim T_{-1}$$

$$A_{[0]_6[2]_2}^{[1-1]_6} = -\frac{1}{2} \sqrt{3} \sum_m (p_m^+ p_m - n_m^+ n_m) \sim T_0,$$

Analytic Formulae for the Matrix Elements...
correspond to the $SU(2)$ generators, which are the components of the pseudospin operator $\hat{T}$. And finally the tensors

\[
A_{[210],2}^{[1]-1} \quad L^M_{11} = \sum_{m,k} C_{1m1k}^{LM} p_m^+ n_k^+,
\]

\[
A_{[210],2}^{[1]-1} \quad L^M_{1-1} = \sum_{m,k} C_{1m1k}^{LM} n_m^+ p_k^+,
\]

and

\[
A_{[210],2}^{[1]-1} \quad L^M_{10} = \frac{1}{\sqrt{2}} \sum_{m,k} C_{1m1k}^{LM} (p_m^+ n_k^+ - n_m^+ p_k^+),
\]

with $L = 1, 2$ and $M = -L, -L + 1, \ldots, L$ extend the $U(3) \times U(2)$ algebra to the $U(6)$ one.

By analogy, the tensors (7) and (8) transform according to \[4\]

\[
[1]_6 \times [1]_6 = [2]_6 + [1, 1]_6.
\]

and

\[
[1]_6^* \times [1]_6^* = [-2]_6 + [-1, -1]_6,
\]

respectively. But, since the basis states of the IVBM are fully symmetric, we consider only the fully symmetric $U(6)$ representations $[2]_6$ and $[-2]_6$ of the operators (7) and (8). Since the two operators $F$ and $G$ are conjugated, i.e., 

\[
(F_{[\lambda]_6}^{[\lambda]_6} \quad L^M_{11} = (-1)^{\lambda+\mu+L-M+T-T_0} G_{[\lambda]_6}^{[\lambda]_6} \quad L^M_{1-1},
\]

where $[\lambda]_3 = (\lambda, \mu)$, we are going to present the next decompositions only for the $F$ tensors (18). According to the decomposition rules for the fully symmetric $U(6)$ irreps [4], we have

\[
[2]_6 = [2]_3 [2]_2 + [1, 1]_3 [0]_2 = (2, 0) [2]_2 + (0, 1) [0]_2,
\]

which further contain in $(2, 0)$ $L = 0, 2$ with $T = 1$ and in $(0, 1) - L = 1$ with $T = 0$. Their explicit expressions in terms of the creation $p_i^+$, $n_i^+$ and annihilation operators $p_i$, $n_i$ at $i = 0, \pm 1$ are

\[
F_{[2]_3}^{[2]_6} \quad L^M_{11} = \sum_{m,k} C_{1m1k}^{LM} p_m^+ n_k^+,
\]

\[
F_{[2]_3}^{[2]_6} \quad L^M_{1-1} = \sum_{m,k} C_{1m1k}^{LM} n_m^+ p_k^+.
\]

\[
F_{[2]_3}^{[2]_6} \quad L^M_{10} = \frac{1}{\sqrt{2}} \sum_{m,k} C_{1m1k}^{LM} (p_m^+ n_k^+ - n_m^+ p_k^+),
\]

and

\[
F_{[1, 1]_3}^{[2]_6} \quad L^M_{00} = \frac{1}{\sqrt{2}} \sum_{m,k} C_{1m1k}^{LM} (p_m^+ n_k^+ + n_m^+ p_k^+).
\]
The above operators and their conjugated ones change the number of bosons by two and realize the symplectic extension of the \( U(6) \) algebra. In this way we have listed all the irreducible tensor operators in respect to the reduction chain (1), that correspond to the infinitesimal operators of the \( Sp(12, R) \) algebra. 

Next, we can introduce the tensor products

\[
T^{(\lambda_1a\lambda_2a)}_{\lambda_3[2T]_2} \omega^{\lambda_3} \frac{L M}{T T_0} = \sum T^{[\lambda_1\lambda_3][2T]_2}_{[\lambda_2]_3} T^{(\lambda_2a)}_{\lambda_3[2T]_2} \omega^{\lambda_3} \frac{L_1 M_1}{T_1(T_0)_{12}} T^{[\lambda_2\lambda_3][2T]_2}_{T_2(T_0)_{12}} \times C^{[\lambda_1|\lambda_3][2T]_2}_{[\lambda_2]_3(T_1)_{22}} C^{[\lambda_2\lambda_3][2T]_2}_{(L_1)_{33}(L_2)_{33}}
\]

(22)

of two tensor operators \( T^{[\lambda_1\lambda_3][2T]_2}_{\lambda_3[2T]_2} \frac{L M}{T T_0} \), which are as well tensors in respect to the considered reduction chain. We use (22) to obtain the tensor properties of the operators in the enveloping algebra of \( Sp(12, R) \), containing the products of the algebra generators. The products of second degree enter the two-body interaction of the phenomenological Hamiltonian of this limit [4]. In this particular case we are interested in the transition operators between states differing by four bosons \( T^{[\lambda_1\lambda_3][2T]_2}_{[\lambda_2]_3[2T]_2} \frac{L M}{T T_0} \), expressed in terms of the products of two operators \( F^{[\lambda_1\lambda_3][2T]_2}_{[\lambda_2]_3[2T]_2} \frac{L M}{T T_0} \). Making use of the decomposition (19) and the reduction rules in the chain (1), we list in Table 1 all the representations of the chain subgroups that define the transformation properties of the resulting tensors.

### Table 1. Tensor products of two raising operators.

<table>
<thead>
<tr>
<th>[2]_1</th>
<th>[2]_2</th>
<th>[4]_1</th>
<th>[O(3)]</th>
<th>[U(2)]</th>
<th>[U(1)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2]_1 [2]_2</td>
<td>[2]_2 [2]_2</td>
<td>[4]_1 [2]_2</td>
<td>0, 0, 2, 4</td>
<td>2</td>
<td>0, 1, ±2</td>
</tr>
<tr>
<td>[2]_2 [2]_2</td>
<td>[2]_2 [0]</td>
<td>[2]_2 [2]_2</td>
<td>1, 2, 3</td>
<td>1</td>
<td>0, ±1</td>
</tr>
<tr>
<td>[0] [0]</td>
<td>[0] [0]</td>
<td>[0] [0]</td>
<td>0, 2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The basis states in the \( \mathcal{H}_+ \) are also obtained as symmetrized tensor products of different degrees of the tensor operators \( F^{[\lambda_1\lambda_3][2T]_2}_{[\lambda_2]_3[2T]_2} \frac{L M}{T T_0} \).

### 3 Construction of the Symplectic Basis States of the IVBM

In order to clarify the role of the tensor operators introduced in the previous section as transition operators and to simplify the calculation of their matrix elements, the basis for the Hilbert space must be symmetry adapted to the algebraic structure along the considered subgroup chain (1). It is evident from (20) and (21), that the basis states of the IVBM in the \( \mathcal{H}_+ \) (\( N \)–even) subspace of the
Thus, in general a basis for the considered dynamical symmetry of the IVBM can be constructed by applying the multiple symmetric coupling (22) of the raising tensors \( F_{\lambda \alpha}^{[2]_6} \) with itself - \([F \times \ldots \times F]^{[\lambda]_6}_{[\lambda]_6} \). Note that only fully symmetric tensor products \( [\lambda]_6 = [N]_6 \) are nonzero, since the raising operator commutes with itself. The possible \( U(3) \) couplings are enumerated by the set \( [\lambda]_3 = \{(n_1, n_2, 0) \equiv (n_1 - n_2, n_2); n_1 \geq n_2 \geq 0 \} \). In terms of the notations \( (\lambda, \mu) \) the \( SU(3) \) content of the \( U(6) \) symmetric tensor \( [N]_6 \) is determined by \( \lambda = n_1 - n_2, \mu = n_2 \). The number of copies of the operator \( F \) in the symmetric tensor product \( [N]_6 \) is \( N/2 \), where \( N = n_1 + n_2 = \lambda + 2\mu \) [3]. Each raising operator will increase the number of bosons \( N \) by two. Then, the resulting infinite basis is denoted by \([|N\rangle(\lambda, \mu); KLM; TT_0 \rangle\), where \( KLM \) are the quantum numbers for the non-orthonormal basis of the irrep \( (\lambda, \mu) \).

The \( Sp(12, R) \) classification scheme for the \( SU(3) \) boson representations, obtained by applying the reduction rules [3] for the irreps in the chain (1) for even value of the number of bosons \( N \), is shown on Table 2. Each row (fixed \( N \)) of the table corresponds to a given irreducible representation of the \( U(6) \) algebra. Then the possible values for the pseudospin, given in the column next to the respective value of \( N \), are \( T = \frac{N}{2}, \frac{N}{2} - 1, \ldots, 0 \). Thus when \( N \) and \( T \) are fixed, \( 2T + 1 \) equivalent representations of the group \( SU(3) \) arise. Each of them is distinguished by the eigenvalues of the operator \( T_0 : -T, -T + 1, ..., T \), defining the columns of Table 2. The same \( SU(3) \) representations \( (\lambda, \mu) \) arise for the positive and negative eigenvalues of \( T_0 \).

Now it is clear which of the tensor operators act as transition operators between the basis states ordered in the classification scheme presented in Table 2. The operators \( F_{[\lambda]_6^{[2]_6}}^{[L,M]}_{[T,T_0]} \) with \( T_0 = 0 \) (21) give the transitions between two neighboring cells \( (\downarrow) \) from one column, while the ones with \( T_0 = \pm 1(20) \) change the column as well \( (\swarrow) \). The tensors \( A_{[2,1]_6^{[2]_6}}^{[1,-1]_6} \) and (13), which correspond to the \( SU(3) \) generators do not change the \( SU(3) \) representations \( (\lambda, \mu) \), but can change the angular momentum \( L \) inside it \( (\Longrightarrow) \). The \( SU(2) \) generating tensors \( A_{[0]_6^{[2]_6}}^{[1,-1]_6} \) (14) change the projection \( T_0 \) \( (\downarrow) \) of the pseudospin \( T \) and in this way distinguish the equivalent \( SU(3) \) irreps belonging to the different columns of the same row of Table 2. Inside a given cell the transition between the different \( SU(3) \) irreps \( (\downarrow) \) is realized by the operators \( A_{[2,1]_6^{[2]_6}}^{[1,-1]_6} \) (15), (16) and (17), that represent the \( U(6) \) generators. The action of the tensor operators on the \( SU(3) \) vectors inside a given cell or between the cells of Table 2, is also schematically presented on it with corresponding arrows. In physical applications sequences
of SU(3) vectors are attributed to sequences of collective states belonging to different bands in the nuclear spectra. By means of the above analysis, the appropriate transition operators can be defined as appropriate combinations of the tensor operators given in Section 2.

4 Matrix Elements of the Transition Operators in Symmetry Adapted Basis

Matrix elements of the $Sp(12, R)$ algebra can be calculated in several ways. A direct method is to use the $Sp(12, R)$ commutation relations [1] to derive recursion relations. Another is to start from approximate matrix element and proceed by successive approximations, adjusting the matrix elements until the commutation relations are precisely satisfied [9]. The third method is to make use of a vector-valued coherent-state representation theory [5], [10] to relate the matrix elements to the known matrix elements of a much simpler Weyl algebra. However, in the present paper we use another technique for calculation of the matrix elements of the $Sp(12, R)$ algebra, based on the fact that the representations of the SU(3) subgroup in IVBM are built with the help of the two kinds of vector bosons, which is in some sense simpler than the construction of the SU(3) representations in IBM [11] and Sp(6, R) symplectic model [12].

In the preceding sections we expressed the $Sp(12, R)$ generators $F_{T T_0}^{LM}$, $G_{T T_0}^{LM}$, $A_{T T_0}^{LM}$ and the basis states as components of irreducible tensors in respect to the reduction chain (1). Thus, for calculating their matrix elements, we have the advantage of using the Wigner-Eckart theorem in two steps. For the $SU(3) \rightarrow SO(3)$ and $SU(2) \rightarrow U(1) \times U(1)$ reduction we need the standard $SU(2)$ Table 2. Classification of the basis states.

<table>
<thead>
<tr>
<th>NT</th>
<th>$T_0 \ \ldots \ 4$</th>
<th>$\pm 3$</th>
<th>$\pm 2$</th>
<th>$\pm 1$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 1</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 1</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>6 2</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>8 2</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1 0</td>
<td></td>
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<td></td>
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<tr>
<td>0 0</td>
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...
Clebsch-Gordan coefficient (CGC)
\[
\langle [N'](\lambda', \mu'); K'L'M'; T'T_0 | T_{[3]}^{[6]} |_{[2]} | T_{[3]}^{[6]} |_{[2]} | T_{[3]}^{[6]} |_{[2]} \rangle_{T_0}
\]
\[= \langle [N'](\lambda', \mu'); K'L' | T_{[3]}^{[6]} |_{[2]} | T_{[3]}^{[6]} |_{[2]} | T_{[3]}^{[6]} |_{[2]} \rangle_{T_0}
\]
\[= \langle [N'](\lambda', \mu'); K'L' | T_{[3]}^{[6]} |_{[2]} | T_{[3]}^{[6]} |_{[2]} | T_{[3]}^{[6]} |_{[2]} \rangle_{T_0}
\]
where \( C_{KL'}^{\lambda, \mu} \) is a reduced \( SU(3) \) Clebsch-Gordan coefficient (CGC), which is known analytically in many special cases \([14] , [15] , [16] , [17] \) and computer codes are available to calculate them numerically \([18] , [19] , [20] \). Hence, for the evaluation of the matrix elements (23) of the \( Sp(12, R) \) operators only their reduced triple-barred matrix elements are required. In order to evaluate them we first obtain the matrix elements of the creation \( u_{m}^{+}(\alpha) \) and annihilation \( u_{m}(\alpha) \) operators \( m = 0, \pm 1; \alpha = \pm \frac{1}{2} \) of the vector bosons that build them. The latter act in the Hilbert space \( \mathcal{H} \) of the boson representation of the algebra of \( Sp(12, R) \) with a vacuum \( |0\rangle \) defined by \( u_{m}(\alpha)|0\rangle = 0 \). In the notations \( u_{i}^{+}(\frac{1}{2}) = p_{i}^{+} \) and \( u_{i}^{-}(\frac{1}{2}) = n_{i}^{+} \), an orthonormal basis in \( \mathcal{H} \) is introduced in the following way \([7]\):
\[
|\pi, \nu\rangle = \prod_{i,k=0,\pm 1} \frac{(p_{i}^{+})^{n_{i}}}{\sqrt{\pi_{i}!}} \frac{(n_{k}^{+})^{\nu_{k}}}{\sqrt{\nu_{k}!}} |0\rangle,
\]
(25)
where \( \pi \equiv \{ \pi_{1}, \pi_{0}, \pi_{-1} \} \) run over the set of three nonnegative numbers for which \( N_{p} = \sum \pi_{i} \), and the same is valid for \( \nu \equiv \{ \nu_{1}, \nu_{0}, \nu_{-1} \} \) with \( N_{n} = \sum \nu_{i} \), where \( N_{p} \) and \( N_{n} \) give the number of bosons of each kind, and the total number of bosons that build each state is \( N = N_{p} + N_{n} \). These numbers are eigenvalues of the corresponding operators \( \hat{N}_{p} = \sum p_{m}^{+} p_{m} \), \( \hat{N}_{n} = \sum n_{m}^{+} n_{m} \), and \( \hat{N} = \hat{N}_{p} + \hat{N}_{n} \):
\[
\hat{N}_{p} |\pi, \nu\rangle = N_{p} |\pi, \nu\rangle, \quad \hat{N}_{n} |\pi, \nu\rangle = N_{n} |\pi, \nu\rangle,
\]
(26)
\[
\hat{N} |\pi, \nu\rangle = N |\pi, \nu\rangle
\]
(27)
As a result of the above relations the basis states \( |\pi, \nu\rangle \) can be labelled with the quantum numbers \( N, N_{p}, N_{n} \). In the considered reduction chain (1) the labels of the \( SU(3) \) - \( (\lambda, \mu) \) irreps are related to the numbers of the introduced \( n \) and \( p \) -vector bosons in the following way \( \lambda = N_{p}, \mu = N_{n}, N = \lambda + 2\mu \) \([3]\). It is simple to see that the eigenvalues in (26) \( N_{p} \equiv n_{1} \) and \( N_{n} \equiv n_{2} \), where \( n_{1}, n_{2} \)
defined the $U(3)$ tensor properties $[\lambda]_3$ in the tensor operators (6), (7) and (8) and the symmetry adapted basis states, given in Table 2. Also the eigenvalue $N$ in (26) corresponds to the totally symmetric $U(6)$ irrep $[N]$ that defines the tensor operators and the basis states tensor properties. Hence the basis (25) can be equivalently labelled by $[[N]; (\lambda, \mu)]$ with $\dim(\lambda, \mu) = \frac{1}{2}(\lambda + \mu + 2)(\lambda + 1)(\mu + 1)$ included in the normalization of the states. The action of any component of the boson creation and annihilation operators is standardly given by

\[ p_+^i [[N]; (\lambda, \mu)] = \sqrt{\frac{(N + 1) \dim(\lambda, \mu)}{\dim(\lambda', \mu')}} [[N + 1]; (\lambda', \mu')] \]

\[ = \sqrt{\frac{(2\mu + 1)(\lambda + \mu + 2)(\lambda + 1)}{(\lambda + \mu + 3)(\lambda + 2)}} [[N + 1]; (\lambda + 1, \mu)], \quad (28) \]

\[ n_+^i [[N]; (\lambda, \mu)] \]

\[ = \sqrt{\frac{(2\mu + 1)(\lambda + \mu + 2)(\lambda + 1)(\mu + 1)}{(\lambda + \mu + 1)(\mu + 2)}} [[N + 1]; (\lambda - 1, \mu + 1)], \quad (29) \]

\[ p_i [[N]; (\lambda, \mu)] = \sqrt{\frac{(2\mu + \lambda)(\lambda + \mu + 2)(\lambda + 1)}{\lambda(\lambda + \mu + 1)}} [[N - 1]; (\lambda - 1, \mu)], \quad (30) \]

\[ n_i [[N]; (\lambda, \mu)] = \sqrt{\frac{(2\mu + \lambda)(\lambda + 1)(\mu + 1)}{(\lambda + 2)(\mu)}} [[N - 1]; (\lambda + 1, \mu - 1)]. \quad (31) \]

With the help of relations (28)–(31) we can evaluate the corresponding matrix elements of the building blocks of the IVBM. They correspond to the triple-barred reduced matrix elements in (24) as they depend only on the $U(6) \rightarrow SU(3)$ quantum numbers. The above expressions for the action of the creation and annihilation operators are very simple and useful, as only a single resulting state is obtained. The $Sp(12, R)$ generators, explicitly presented in Section 2 as tensor operators in terms of bilinear products of $p_+^i, n_+^i, p_i$ and $n_i$, can be considered as coupled $U(6) \rightarrow SU(3)$ tensors (22) and their matrix elements calculated using the above expressions where the state resulting from the action of the operators taken as an intermediate state [13].

5 Conclusions

In the present paper we investigate the tensor properties of the algebra generators of $Sp(12, R)$ with respect to the reduction chain (1). $Sp(12, R)$ is the group
of dynamical symmetry of the IVBM and the considered chain of subgroups was applied in [3] for the description of positive and negative parity bands in well deformed nuclei. The basis states of the model Hamiltonian are also classified by the quantum numbers corresponding to the irreducible representations of its subgroups and in this way the symmetry adapted basis in this limit of the IVBM is constructed. The action of the symplectic generators as transition operators between the basis states is presented. Simple analytical expressions for the matrix elements of the basic building blocks of the model are obtained as well. Making use of the latter and respective generalization of the Wigner-Eckart theorem one is able to evaluate explicitly the matrix elements of the transition operators expressed in terms of tensor operators. By definition such matrix elements give the transition probabilities between the collective states attributed to the basis states of the Hamiltonian. In this way useful mathematical tool is developed, which will allow future applications of the symplectic IVBM in the description of different collective features of the nuclear systems. Furthermore, we hope that such investigations will contribute for deeper understanding of the physical meaning of the mathematical structures of the model and more correct evaluation of its limits of applicability.

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References

Analytic Formulae for the Matrix Elements...