Bianchi III String Cosmology in Lyra Geometry

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Abstract. A spatially homogeneous Bianchi type III cosmological solutions of massive strings are obtained within the framework of Lyra geometry. The equations of state for strings have been used for different solutions. The physical implications of the models are briefly discussed.

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1 Introduction

In last two decades, the study of cosmic strings has received considerable interest as they are believed to give rise to density perturbations leading to formation of galaxies [1-5]. It is noted that the present day observations do not rule out the possible existence of large-scale networks of strings in the early universe. These strings possess stress energy and are coupled to the gravitational field. Several authors have studied the gravitational effects such strings [1-5]. At first, Letelier and Stachel [1] have developed the general relativistic treatment of strings by introducing the energy momentum tensor

\[ T_{ik} = \rho U_i U_k - \lambda W_i W_k, \]

\[ U_i U^i = -W_i W^i = 1, \]

\[ W^i U_i = 0, \]

as the source term in the Einstein field equations. 

\( T_{ik} \) in (1) represents the energy momentum tensor associated with a cloud of strings with particles attached to them. \( \rho \) and \( \lambda \) respectively denote the energy density and the string tension density of the string cloud which are related by

\[ \rho = \rho_p + \lambda \]

where \( \rho_p \) is the particle density in the string cloud.

The energy conditions imply \( \rho \geq 0, \rho_p \geq 0 \), leaving the sign of the string tension density \( \lambda \) unrestricted. The unit time-like vector \( U^i \) is the flow vector of the
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matter and the space-like vector $W^i$, specifies the string direction of the cloud. Exact solution of string cosmology, in the context of space-times of Bianchi types have been obtained by several authors [5].

Since the discovery of general relativity by Einstein, there have been numerous modifications of it. Lyra [6] proposed a modification of Riemannian geometry by introducing a gauge function into the structure manifold that bears a close resemblance to Weyl’s geometry. In consecutive investigations Sen [7] and Sen and Dunn [7] proposed a new scalar tensor theory of gravitation and constructed an analog of the Einstein field equation based on Lyra’s geometry which in normal gauge may be written as

$$R_{ik} - \frac{1}{2} g_{ik} R + \frac{3}{2} \phi_i \phi_k - \frac{3}{4} g_{ik} \phi_m \phi^m = -8\pi G T_{ik}$$  \hspace{1cm} (5)

where $\phi_i$ is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

According to Halford [8], the present theory predicts the same effects within the observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearized form of the field equations. Subsequent investigations were done by several authors in scalar tensor theory and cosmology within the framework of Lyra geometry [7-8]. However, Soleng [9] has pointed out that the cosmologies based on Lyra’s manifold with constant gauge vector $\phi_i$ will either include a creation field and be equal to Hoyle’s creation field cosmology or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term.

We consider in this paper Bianchi III space-times in the context of string cosmology within the framework of Lyra geometry and obtain some exact solutions of string cosmology.

2 The Basic Equations

The ansatz for the spatially homogeneous Bianchi III space-time is taken in the following form as

$$ds^2 = dt^2 - A^2 dx^2 - A^2 e^{-2ax} dy^2 - C^2 dz^2$$  \hspace{1cm} (6)

where $A, C$ are functions of time alone and $a$ is a constant.

The displacement vector is taken as

$$\phi_i = (\beta(t), 0, 0, 0).$$  \hspace{1cm} (7)

In a co-moving co-ordinate system we have

$$U^i = (0, 0, 0, 1)$$  \hspace{1cm} (8)

$$W^i = (0, 0, (1/C), 0)$$  \hspace{1cm} (9)
The field equation (5) for the metric (6) leads to the following system of equations:

\[
\begin{align*}
2[A^1 C^1 / AC] + \left( [A^1]^2 / A^2 \right) - \left[ a^2 / A^2 \right] - (3/4) \beta^2 &= \rho \\
2[A^{11} / A] + \left( [A^1]^2 / A^2 \right) + \left[ a^2 / A^2 \right] + (3/4) \beta^2 &= \lambda \\
[A^{11} / A] + [C^{11} / C] + [A^1 C^1 / AC] + (3/4) \beta^2 &= 0
\end{align*}
\]

(‘\*\*’ denotes differentiation w.r.t. ‘t’)

The expansion scalar \( \theta \) and the shear scalar \( \sigma \) respectively found to have the following expressions:

\[
\begin{align*}
\theta &= \left( [C^1 / C] + 2(A^1 / A) \right) \\
\sigma^2 &= (2/3)[(C^1 / C) - (A^1 / A)]^2
\end{align*}
\]

The different equations of state for string model be \([1-5]\)

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<th>Description</th>
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In the following section, we shall determine the exact solution of field equations using above equations of state for string model in Lyra geometry.

### 3 Solutions

#### Case I: Barotropic equation of state

In this case we take displacement vector as a constant, i.e.,

\[ \beta = \text{constant} \]

The field equations (10)–(12) are a system of three equations with four unknown parameters \( \rho, \lambda, A, \) and \( C \). An additional constraint relating these parameters is required to obtain explicit solutions of the system.

Here we assume the relation

\[ A = \mu C^n \]

(\( \mu, n \) are constants) between the scale factors for unique solution of the field equations.

Using this relation, we get from (12) as:

\[ C = \left[ k(t) \right]^{1/(1+d)} \]
where
\[ k(t) = [D(d + 1)/h]^{1/2} \sin \sqrt{(d + 1)/h(t - t_0)} \] (20)
and \( h = [3\beta^2/(n + 1)], d = [n^2/(n + 1)], t_0 \) and \( D \) are integration constants.

The other physical parameters have the following expressions:

\[ A = \mu[k(t)]^{n/(1+d)} \] (21)
\[ \theta = (2n + 1)R(t) \] (22)
\[ \sigma^2 = (2/3)(n - 1)^2 R^2(t) \] (23)
\[ \rho = (n^2 + 2n)R^2(t) - a^2 \mu^{-2}[k(t)]^{-[2n/(1+d)]} - (3/4)\beta^2 \] (24)
\[ \lambda = (3n^2 - 2n - 2dn)R^2(t) - a^2 \mu^{-2}[k(t)]^{-[2n/(1+d)]} - 2nh + (3/4)\beta^2 \] (25)
\[ \rho_p = (4n + 2dn - 2n^2)R^2(t) - (3/2)\beta^2 + 2nh \] (26)

where
\[ R^2(t) = D\cos^2 \sqrt{(d + 1)/h(t - t_0)}. \] (27)

The above expressions indicate that the physical and kinematical parameters diverge for \( t = t_0 \). Accordingly the solution represents a Bianchi–III string model in Lyra geometry beginning with the singularity \( t = t_0 \).

The expansion parameter \( \theta \) is positive and the expansion will cease at the epoch
\[ t = t_0 + \frac{\pi}{2(d + 1)/h}. \]

At this epoch, we have \( \lambda = 0 \) and \( \rho = \rho_p \), provided
\[ 3/4\beta^2 = 2nh + a^2 \mu^{-2}[h/D(d + 1)]^{[n/(1+d)]} \] (28)

Therefore at this instant strings vanish and we are left with a dust filled universe.

At this time,
\[ \rho = 2nh - (3/2)\beta^2 \] (29)
\[ \theta = \sigma^2 = 0 \] (30)
\[ A = \mu[\{D(d + 1)/h\}^{n/2(1+d)}], \quad C = [\{D(d + 1)/h\}^{1/2(1+d)}] \] (31)

So all the parameters are of finite magnitude. Moreover we note that anisotropy and expansion rate vanish at this moment.

**Case II: Geometric string (\( \rho = \lambda \))**

Here we also assume the same relation between the metric coefficients, i.e., \( A = \mu C^n \) but the displacement vector is not constant.
Following the relation (10) - (11) + 2.(12) and using the above polynomial relation between the metric coefficients, we get

\[ \frac{C^{11}}{C} + 2n\left(\frac{C^{1}C^{2}}{C} \right)^2 = 0. \]  

(32)

Solving this equation, we get

\[ C = K_3[t - t_0]^{1/(2n+1)} \]  

(33)

where \( t_0, K_3 \) are integration constants.

So the metric coefficient and other physical quantities are obtained as

\[ A = \mu[K_3]^n[t - t_0]^{n/(2n+1)} \]  

(34)

\[ \rho = -a^2\mu^{-2}[t - t_0]^{-2n/(2n+1)} \]  

(35)

\[ (3/4)\beta^2 = \left(\frac{(2n + n^2)}{(2n + 1)^2}\right)[t - t_0]^{-2} \]  

(36)

\[ \theta = [t - t_0]^{-1} \]  

(37)

\[ \sigma^2 = \frac{2(n - 1)^2/3(2n + 1)^2}{[t - t_0]^{-2}}. \]  

(38)

This model is not interesting because it violates the positivity of energy condition.

Case III: Takabayasi string (i.e., P-string)

Here the equation of state \( \rho = (1 + w)\lambda \), where \( w > 0 \), a constant, and it is small for string dominant era and large for particle dominant era.

Further using the polynomial relation \( A = \mu C^n \), from the field equation, we get

\[ \int \left(\frac{k_2}{2I + 2n + 2}\right)C^{2-2n} + D_1C^{-2I} \right)^{-1/2} dC = \pm(t - t_0) \]  

(39)

where \( I = [(2wn2 - 4n - 2wn)/2 + w - nw] \), \( k_2 = [(wa^2\mu^{-2})/(2 + w - nw)] \) and \( D_1, t_0 \) are integration constants.

This is a complicated integral equation containing so many arbitrary parameters. The equation can be solved only when the integration constant \( D_1 \) is zero.

Hence we get

\[ C = [n\sqrt{k_2}(t - t_0)]^{1/n}. \]  

(40)

The other parameters are,

\[ A = \mu n\sqrt{k_2}(t - t_0) \]  

(41)

\[ \rho = [(n^2 + 2n + 1)/n^2 - a^2/(\mu^2n^2k_2^2)][t - t_0]^{-2} \]  

(42)

\[ \theta = [2 + (1/n)](t - t_0)^{-1} \]  

(43)

\[ \sigma^2 = (2/3)[(1/n) - 1]^2(t - t_0)^{-2} \]  

(44)

\[ (3/4)\beta^2 = -[4/3n^2][t - t_0]^{-2} \]  

(45)
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In this case we see that $C$ takes the imaginary values. So, either in Bianchi III model, the Takabayasi string is not possible or the manifold with imaginary connection has been considered. If the latter is true, the physical significance of such an imaginary quantity is not very clear. Perhaps, a more convenient way of looking at this difficulty is to let $-\beta^2 = \Lambda(t)$ (say), which is real and which can then be thought of as a cosmological term.

In this case, we see that $t = t_0$ is the initial epoch and it is a point singularity. At this instant $\theta, \sigma^2 \to \infty$. Thus the Universe starts with an infinite rate of expansion and measure of anisotropy. At $t$ gradually increases $\theta, \sigma^2$ decreases and finally when $t \to \infty$, $\theta, \sigma^2 \to 0$.

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References
