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Remarks on Yamabe Soliton and Gradient Yamabe Soliton

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Abstract. The paper examines two main topics: first, properties of the Yamabe soliton are explored by discussing a parallel vector field and demonstrating the validity of the Killing equation under curvature collineation. Second, the study of the gradient Yamabe soliton involves showing the skew-symmetry of the first two vector fields of the Hessian curvature tensor and deriving the soliton equation's expression in affine collineation.

KEY WORDS: Yamabe soliton, parallel vector field, Curvature collineation, Killing equation, Gradient Yamabe soliton, Proper homothetic vector field, Affine collineation, Einstein manifold.

1 Introduction:

In modern geometry, semi-Riemannian manifolds are essential, finding extensive applications in relativity and cosmology within applied physics. The historical context begins with Hamilton [1,2], who introduced the Yamabe flow in 1988 to study Yamabe's conjecture: that an arbitrary metric can be conformally transformed into a metric possessing constant scalar curvature. In this framework, Yamabe solitons represent self-similar solutions to the flow equation.

A semi-Riemannian manifold is termed a Yamabe soliton [1] if its metric (g) evolves according to the vector field V as:

$$L_V g = (\lambda - r)g, \tag{1}$$

where L_V is the Lie derivative, r is the scalar curvature, and λ is a real constant. A Yamabe soliton is considered a gradient Yamabe soliton [2] if the vector field V is the gradient of a smooth function F (i.e., $V = \nabla F$), and this function (often called the Hessian function) satisfies the defining condition:

$$\nabla_X \nabla_Y F = (r - \lambda)g(X, Y), \tag{2}$$

or any vector fields X and Y, with ∇ being the Levi-Civita connection.

Solitons have been extensively studied, with major contributions and explanations provided by many authors, notably Hamilton [1], Barman [3–5], Duggal and Sharma [6], Erken [7], O'Neill [8], Ozen [9] and many others.

The organization of this paper is as follows: After the Introduction, Section 2 analyzes the Yamabe soliton by determining the form of its scalar curvature in semi-Riemannian manifolds subject to specific constraints, including the proper homothetic condition and coupling with an Einstein manifold via curvature collineations. Subsequently, Section 3 investigates the Gradient Yamabe soliton, demonstrating the skew-symmetry of the curvature tensor's action on the Hessian function and establishing the soliton equation in the presence of an affine collineation vector field.

2 Yamabe Soliton

Theorem 1. *If the vector field V defining a Yamabe soliton is parallel, then the scalar curvature r is equal to the constant λ defining the soliton.*

Proof. Considering equation (1), it is established that:

$$L_V g(X,Y) = (\lambda - r)g(X,Y). \tag{3}$$

In general, the Lie derivative of the metric is

$$L_V g(X,Y) = \nabla_X VY + \nabla_Y VX. \tag{4}$$

Substituting (4) into (3) yields:

$$\nabla_X VY + \nabla_Y VX = (\lambda - r)g(X,Y). \tag{5}$$

If the vector field V on a semi-Riemannian manifold is parallel [10] with respect to the Levi-Civita connection ∇, it means:

$$\nabla_X V = \nabla_Y V = 0. \tag{6}$$

Using (5) and (6), we conclude:

$$\lambda = r.$$

Here proved the theorem.

□

Definition 1. A vector field V of a 4-dimensional semi-Riemannian manifold is proper homothetic [6], defined by:

$$L_V g(X,Y) = 2cg(X,Y), \tag{7}$$

where c be a non-zero constant.

Theorem 2. *On a* 4*-dimensional proper homothetic Yamabe soliton of a semi-Riemannian manifold, the scalar curvature* r *satisfies* $r = \lambda - 2c$*.*

Proof. A direct substitution of (7) into (1) results in:

$$(\lambda - r)g(X,Y) = 2cg(X,Y). \tag{8}$$

Rearranging (8) gives:

$$r = \lambda - 2c.$$

This proves the theorem. ☐

Definition 2. A vector field V on a semi-Riemannian manifold is a curvature collineation [6] if it satisfies the condition that the Lie derivative of the Ricci tensor S with respect to V is zero:

$$L_V S(X,Y) = 0. \tag{9}$$

Definition 3. If the Ricci tensor S on semi-Riemannian manifold is given by

$$S(X,Y) = kg(X,Y), \tag{10}$$

where k is a constant, then semi-Riemannian manifold is called an Einstein manifold [11].

Theorem 3. *If a semi-Riemannian manifold is a Yamabe soliton and possesses a vector field* V *that is both a curvature collineation* $L_V S(X,Y) = 0$ *and defined on an Einstein manifold* $(S(X,Y) = kg(X,Y))$*, where the Einstein constant* k *is non-zero* $(k \neq 0)$ *:*

(i) The vector field V is a Killing vector field; that is, the manifold satisfies the Killing equation $(L_V g = 0)$*.*

(ii) The scalar curvature r *of the manifold is equal to the soliton constant* $\lambda (r = \lambda)$*.*

Proof. Combining equations (9) and (10) yields:

$$L_V g(X,Y) = 0. \tag{11}$$

The resulting equation is known as the Killing equation, demonstrating that V is a Killing vector field.

Substituting Equation (11) into the Yamabe soliton Equation (1) yields

$$r = \lambda.$$

We conclude the theorem here. ☐

Lemma 1. *[6]. The Killing equations show that if V is a Killing vector field then the local geometry remains invariant while moving along the local 1-parameter group of local transformations ϕ_t generated by V.*

From the above theorem and lemma, we can evaluate that

Theorem 4. *If a semi-Riemannian manifold's curvature collineations are coupled to the Einstein manifold $(S = kg)$ on the Yamabe soliton and the Einstein constant k is non-zero $(k \neq 0)$, then the local geometry remains invariant under the local 1-parameter group of transformations ϕ_t generated by the vector field V.*

3 Gradient Yamabe Soliton

Theorem 5. *On a gradient Yamabe soliton, the vector field defined by the curvature tensor acting on the Hessian function, $R(X, Y)F$, is skew-symmetric in X and Y.*

Proof. Due to the symmetry of the metric g, interchanging X and Y in equation (2) gives

$$\nabla_Y \nabla_X F = (r - \lambda)g(X, Y). \tag{12}$$

We use the general identity relating the curvature tensor R to the covariant derivatives of a function F:

$$R(X, Y)F = \nabla_X \nabla_Y F - \nabla_Y \nabla_X F - \nabla_{[X,Y]}F. \tag{13}$$

We rearrange equation (13) by isolating the difference of covariant derivatives:

$$R(X, Y)F + \nabla_{[X,Y]}F = \nabla_X \nabla_Y F - \nabla_Y \nabla_X F. \tag{14}$$

Substituting equations (2) and (12) into equation (14) yields

$$R(X, Y)F + \nabla_{[X,Y]}F = 0.$$

We conclude from the preceding equation that

$$R(X, Y)F = -\nabla_{[X,Y]}F. \tag{15}$$

The identity (15) can be re-expressed using the skew-symmetry of the Lie bracket as

$$R(Y, X)F = \nabla_{[X,Y]}F. \tag{16}$$

By substituting Equation (16) into Equation (15), we obtain

$$R(X, Y)F = -R(Y, X)F. \tag{17}$$

The theorem is proved. □

Theorem 6. *On a gradient Yamabe soliton, if the vector fields X and Y commute* $([X, Y] = 0)$, *then the curvature tensor acting on the Hessian function* F *vanishes identically:* $R(X, Y)F = 0$. *The manifold is flat if and only if its sectional curvature is identically zero.*

If $[X, Y] = 0$, then from the Equation (15), it implies that

$$R(X, Y)F = 0. \tag{18}$$

Consequently, the manifold satisfies the criterion for flatness, meaning the gradient Yamabe soliton is flat.

Proposition 1. *[8]. A semi-Riemannian manifold is flat iff the sectional curvature is identically zero.*

Applying Proposition 1 (which states that a manifold is flat if and only if the sectional curvature is identically zero) to the result in equation (18), we conclude the theorem.

Definition 4. A vector field V on a manifold with a symmetric affine connection (∇) is called an affine collineation [6] if the Lie derivative of the connection (∇) with respect to V vanishes

$$L_V \nabla = 0. \tag{19}$$

Theorem 7. *If* V *is an affine collineation* $(L_V \nabla = 0)$ *on a semi-Riemannian manifold that is a gradient Yamabe soliton and the scalar curvature* r *equals the soliton constant* $\lambda (r = \lambda)$, *then the following condition holds for the Hessian function* F*:*

$$\nabla_X \nabla_Y (L_V F) = 0.$$

Proof. Taking the Lie derivative L_V on both sides of the Gradient Yamabe soliton equation (2) and applying the Lie derivative rules to the covariant derivatives, we obtain the expression

$$\begin{aligned}(L_V \nabla_X)(\nabla_Y F) + \nabla_X (L_V \nabla_Y)F + \nabla_X \nabla_Y (L_V F) \\ = (r - \lambda)(L_V g)(X, Y).\end{aligned} \tag{20}$$

Using the definition of an affine collineation (equation (19)), to simplify equation (20), we find that

$$\nabla_X \nabla_Y (L_V F) = (r - \lambda)(L_V g)(X, Y). \tag{21}$$

Setting $r = \lambda$ in equation (21), we conclude that:

$$\nabla_X \nabla_Y (L_V F) = 0.$$

This completes the proof of the theorem.

□

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